WAVELET TAYLOR GALERKIN SHOCK WAVE TUBE PROBLEM

# WTG × SWT

Application of **Wavelet Taylor Galerkin** method to **1D Euler Equation**

YAMAMOTO KYOSUKE
Asst. Prof.
University of Tsukuba

MATSUURA TAISHI
graduated from
University of Tsukuba

We want to make

**C**omputational **F**luid **D**ynamics

Fast

Stable

Accurate

# WTG method is an answer

for limited models of which nodes are regularly aligned

# Euler equation

$$\frac{\partial}{\partial t}\left\{\begin{array}{c}\rho\\ \rho u\\ e\end{array}\right\}+\frac{\partial}{\partial x}\left\{\begin{array}{c}\rho u\\ p+\rho u^2\\ (e+p)\,u\end{array}\right\}=\left\{\begin{array}{c}0\\ 0\\ 0\end{array}\right\}$$

density: $\rho$ pressure: $p$

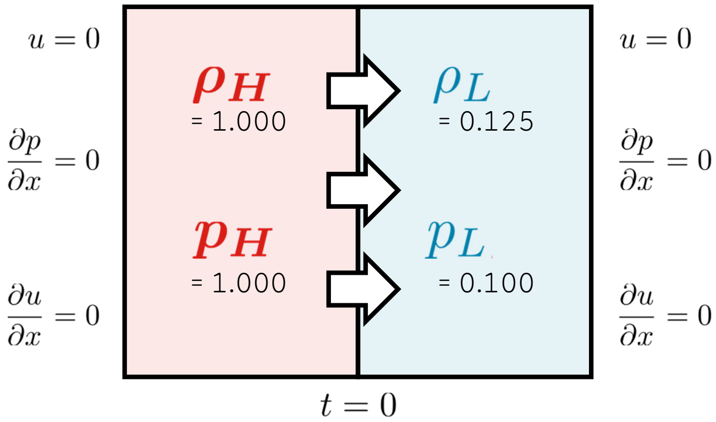
flux: $u$ energy: $e$

$p=(\gamma-1)\left(e-\frac{1}{2}\rho u^2\right)$

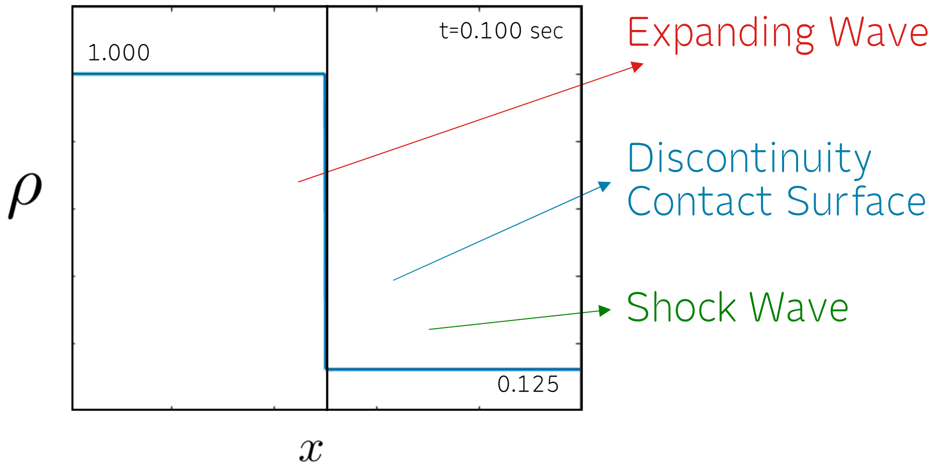
**Euler equation**

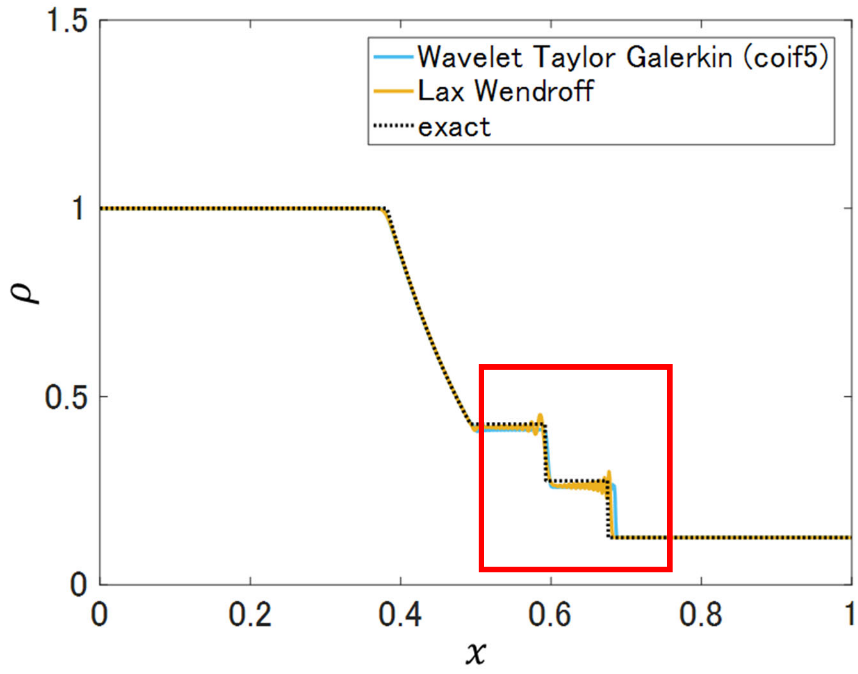
$$\frac{\partial}{\partial t}\left\{\begin{array}{c}\rho\\ \rho u\\ e\end{array}\right\}+\frac{\partial}{\partial x}\left\{\begin{array}{c}\rho u\\ p+\rho u^2\\ (e+p)\,u\end{array}\right\}=\left\{\begin{array}{c}0\\ 0\\ 0\end{array}\right\}$$

can be used for **Shock Wave Tube**



# Analytical solution of **SWT**




1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
$\rho$
$x$
Wavelet Taylor Galerkin (coif5)
Lax Wendroff
exact

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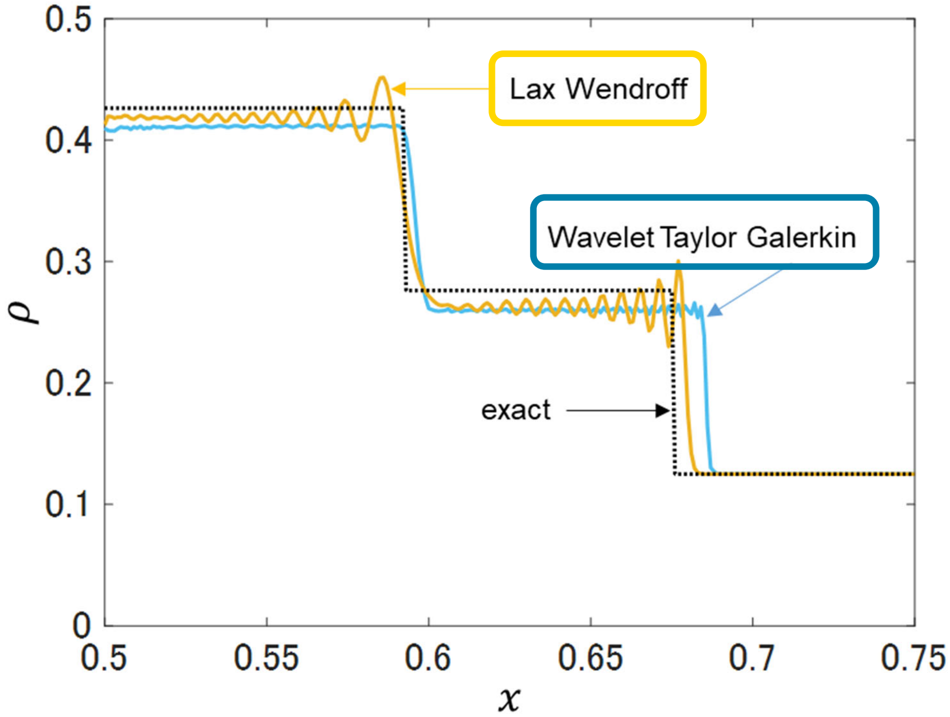
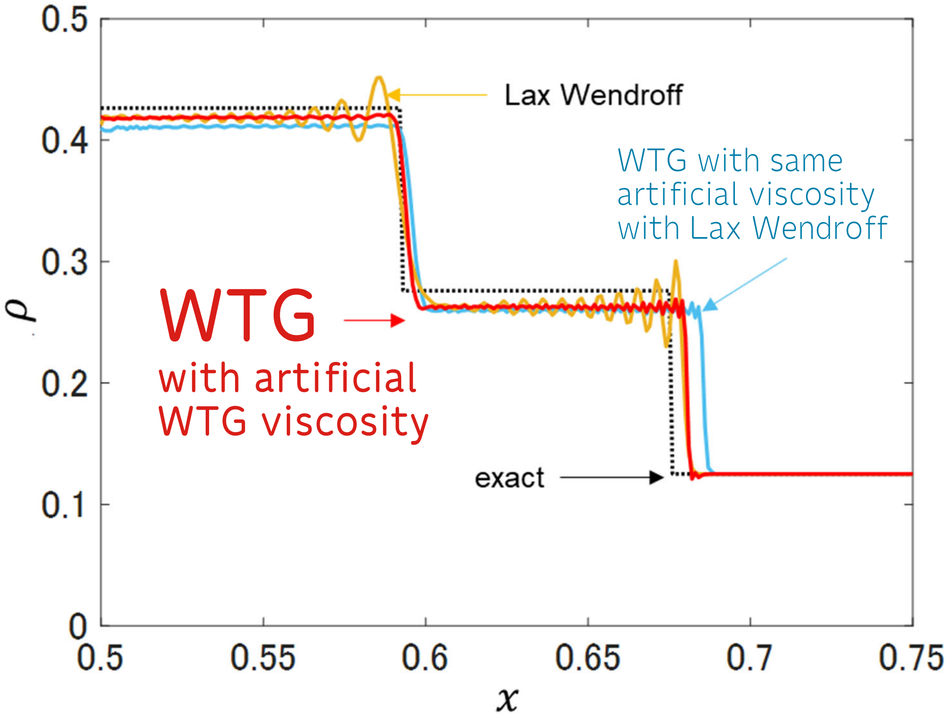



Good:
WTG’ s **amp. of error vibration**
is smaller than LW
(popular FDM)

Negative:
WTG’ s **shock wave propagation**
is much faster
than LW and **Exact Solution**
(popular FDM)




If you calculate the $\left[\frac{\partial^2}{\partial x^2}\right]$ term based on Wavelet function,

WTG shows **better accuracy** for **density** and **propagation time**

## Conclusion

**WTG** is superior to FDM
in terms of accuracy

## Discussion

# How to apply WTG to general models

WTG' s Limitation:
**Nodes should be regularly aligned**

WTG

- **Wavelet** indicates **Discretization** by scaling function
- **Taylor** indicates **Taylor Expansion** for higher accuracy
- **Galerkin** indicates **Simultanization** based on weighted residual

Discretization is

# Interpolation

$$u(t,x) = \boldsymbol{u} \cdot \boldsymbol{N}(t,x)$$

numerical solution    basis function

converting continuous functions into a numerical sequence

(solution of differential equation)    (solution values at discrete points

Taylor Expansion is

# Time Discretization

$$\frac{\partial \boldsymbol{Q}}{\partial t} + \frac{\partial \boldsymbol{E}}{\partial x} = \mathbf{0}$$

$$\boldsymbol{R} = \boldsymbol{Q}^{n+1} - \boldsymbol{Q}^{n} + \Delta t \frac{\partial \boldsymbol{E}}{\partial x}^{n} - \frac{\Delta t^2}{2} \frac{\partial}{\partial x} \mathbf{A}^{n} \frac{\partial \boldsymbol{E}}{\partial x}^{n}$$



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Simultanization is

# Weighted Residual Method

$$\int \boldsymbol{\omega}^{\mathrm{T}} \boldsymbol{R} \mathrm{d}x = 0$$

Residual is a given Governing Equation

Galerkin method is a kind of WRM

$$\frac{\partial \boldsymbol{Q}}{\partial t} + \frac{\partial \boldsymbol{E}}{\partial x} = \boldsymbol{0}$$

**Discretization**

$$\boldsymbol{Q} = \begin{Bmatrix} \rho \\ \rho u \\ e \end{Bmatrix} = \begin{bmatrix} \boldsymbol{N}^{\mathrm{T}} & & \\ & \boldsymbol{N}^{\mathrm{T}} & \\ & & \boldsymbol{N}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\rho u} \\ \boldsymbol{e} \end{Bmatrix}$$

**Wavelet**

$\boldsymbol{N}$ is scaling function

**Time Discretization = Taylor Equation**

$$\boldsymbol{R} = \boldsymbol{Q}^{n+1} - \boldsymbol{Q}^{n} + \Delta t \frac{\partial \boldsymbol{E}^{n}}{\partial x} - \frac{\Delta t^2}{2} \frac{\partial}{\partial x} \mathbf{A}^{n} \frac{\partial \boldsymbol{E}^{n}}{\partial x}$$

**Simultanization**

$$\int \boldsymbol{\omega}^{\mathrm{T}} \boldsymbol{R} \mathrm{d}x = 0$$

**Galerkin Method**

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{N}^{\mathrm{T}} & & \\ & \boldsymbol{N}^{\mathrm{T}} & \\ & & \boldsymbol{N}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_3 \end{Bmatrix}$$

$$\frac{\partial \boldsymbol{Q}}{\partial t} + \frac{\partial \boldsymbol{E}}{\partial x} = \boldsymbol{0}$$

Taylor Galerkin

$$\boldsymbol{Q} = \begin{Bmatrix} \rho \\ \rho u \\ e \end{Bmatrix} = \begin{bmatrix} \boldsymbol{N}^{\mathrm{T}} & & \\ & \boldsymbol{N}^{\mathrm{T}} & \\ & & \boldsymbol{N}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\rho} \\ \boldsymbol{\rho u} \\ \boldsymbol{e} \end{Bmatrix} \qquad \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{bmatrix} \boldsymbol{N}^{\mathrm{T}} & & \\ & \boldsymbol{N}^{\mathrm{T}} & \\ & & \boldsymbol{N}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_3 \end{Bmatrix}$$

$$\int \boldsymbol{\omega}^{\mathrm{T}} \boldsymbol{R} \mathrm{d}x = 0 \qquad \boldsymbol{R} = \boldsymbol{Q}^{n+1} - \boldsymbol{Q}^{n} + \Delta t \frac{\partial \boldsymbol{E}}{\partial x}^{n} - \frac{\Delta t^2}{2} \frac{\partial}{\partial x} \mathbf{A}^{n} \frac{\partial \boldsymbol{E}}{\partial x}^{n}$$

⇩

$$\mathbf{M}\boldsymbol{Q}^{n+1} = \boldsymbol{b}^{n}$$

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$$\mathbf{M} = \begin{bmatrix} \int \boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}\mathrm{d}x & & \\ & \int \boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}\mathrm{d}x & \\ & & \int \boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}\mathrm{d}x \end{bmatrix}$$

$$\boldsymbol{b}^n = \left\{ \begin{array}{l} \int \left\{ \boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{Q}^n + \left( E_1^n - \frac{\partial A_{11}}{\partial x} - \frac{\partial A_{12}}{\partial x} - \frac{\partial A_{13}}{\partial x} \right) \frac{\partial \boldsymbol{N}}{\partial x} \right\} \mathrm{d}x \\ \int \left\{ \boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{Q}^n + \left( E_1^n - \frac{\partial A_{21}}{\partial x} - \frac{\partial A_{22}}{\partial x} - \frac{\partial A_{23}}{\partial x} \right) \frac{\partial \boldsymbol{N}}{\partial x} \right\} \mathrm{d}x \\ \int \left\{ \boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{Q}^n + \left( E_1^n - \frac{\partial A_{31}}{\partial x} - \frac{\partial A_{32}}{\partial x} - \frac{\partial A_{33}}{\partial x} \right) \frac{\partial \boldsymbol{N}}{\partial x} \right\} \mathrm{d}x \end{array} \right\}$$

can be calculated from $\boldsymbol{Q}^n$ and $\boldsymbol{N}$

# The merit of WTG

If you use


Scaling Function

as $\boldsymbol{N}$

$$\mathbf{M}\boldsymbol{Q}^{n+1} = \boldsymbol{b}^n$$

becomes

$$\boldsymbol{Q}^{n+1} = \boldsymbol{b}^n$$

$$\mathbf{M}\boldsymbol{Q}^{n+1} = \boldsymbol{b}^n$$

⇩

When the interval of nodes is fixed,

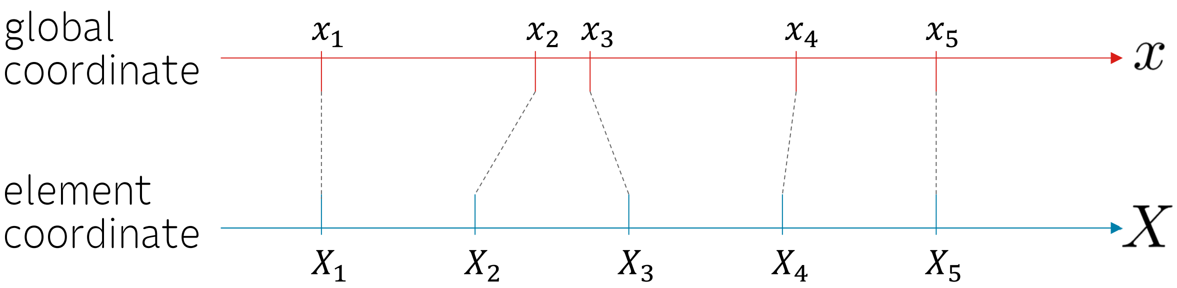
$$\int N_i N_j \mathrm{d}x = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \Rightarrow \mathbf{M} = \mathbf{I}$$

⇩

$$\boldsymbol{Q}^{n+1} = \boldsymbol{b}^n$$

If the nodes' intervals are not fixed,
we have to find

$$\int N_i N_j \frac{\partial x}{\partial X} \mathrm{d}X = \int N_i N_j \frac{\partial \boldsymbol{x}^{\mathrm{T}} \boldsymbol{N}}{\partial X} \mathrm{d}X = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$



# Conclusion

WTG is a good scheme to numerically solve CFD

If you want to apply WTG to general models,
you must find a scheme to find a function satisfying

$$\int N_i N_j \frac{\partial \boldsymbol{x}^{\mathrm{T}} \boldsymbol{N}}{\partial X} \mathrm{d}X = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$