Large Deformation Problem of Solid

Application of Finite Element Method

Phenomena such as **failure** & **breakage**

are often required to be **simulated**

Failure & Breakage = **Large Deformation**

Solid fracture often occurs during large deformation

A very popular numerical simulation method

Finite Element Method

Equation of Equilibrium

$$\int_S \boldsymbol{t}\,\mathrm{d}a + \int_B \boldsymbol{b}\,\mathrm{d}v = \boldsymbol{0}$$

Finite Element Method

$$\int_B \boldsymbol{w}\cdot\boldsymbol{r}\,\mathrm{d}v = 0$$

Equation of Equilibrium
for Large Deformation Problem

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\quad \mathbf{B} \quad \right] \left\{ \boldsymbol{S} \right\} \mathrm{d}V$$

Equation of Equilibrium for Large Deformation Problem

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\ \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\ \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \end{bmatrix} \begin{Bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ s_{12} \\ s_{23} \\ s_{31} \end{Bmatrix} \mathrm{d}V$$

Equation of Equilibrium
for Large Deformation Problem

$$
\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\begin{array}{ccc:ccc}
\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z}
\end{array}\right] \begin{Bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ \hdashline s_{12} \\ s_{23} \\ s_{31} \end{Bmatrix} \mathrm{d}V
$$

Equation of Equilibrium
for Large Deformation Problem

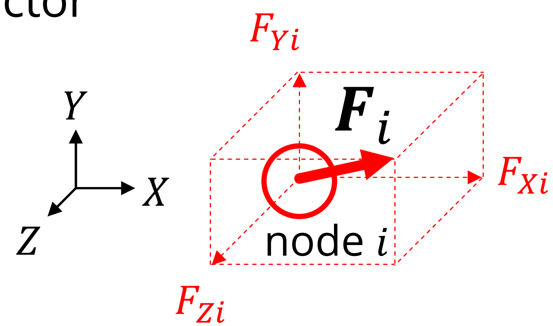
$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\begin{array}{ccc|ccc}
\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z}
\end{array}\right] \begin{Bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ \hline s_{12} \\ s_{23} \\ s_{31} \end{Bmatrix} \mathrm{d}V$$

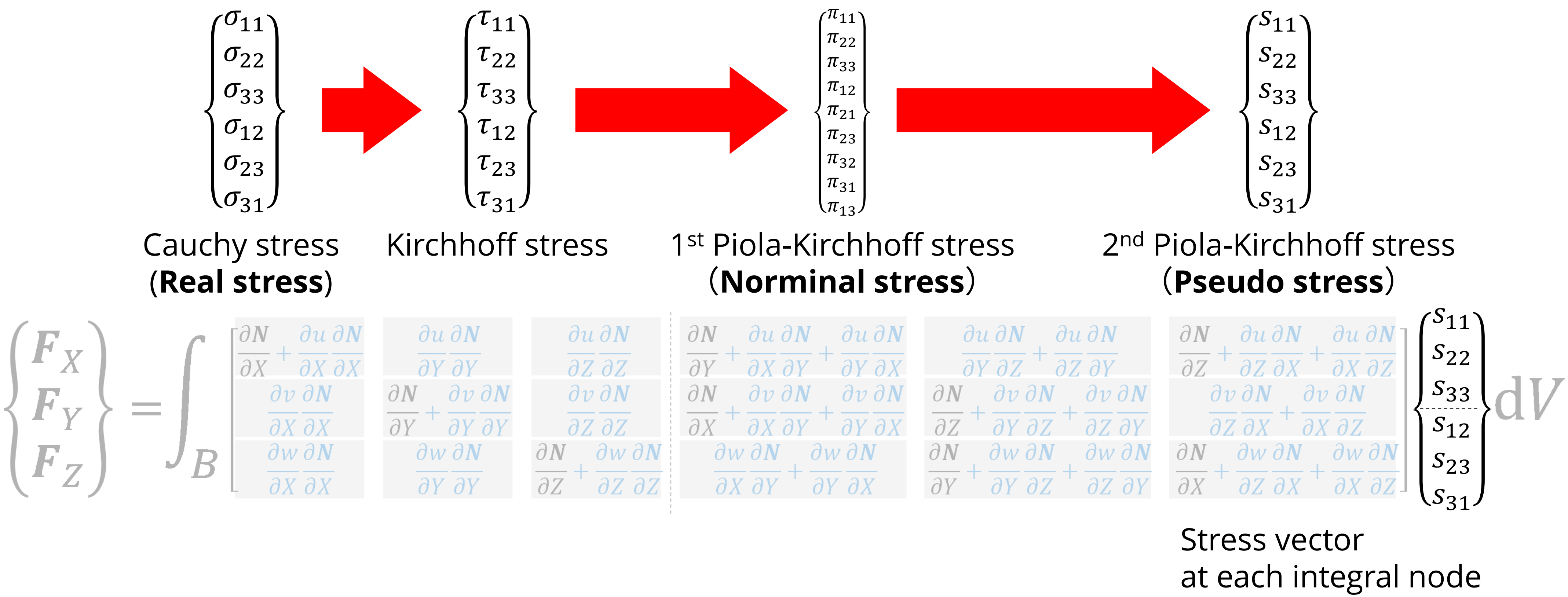
Degree of Freedom

$$\boldsymbol{F}_X = \begin{Bmatrix} \vdots \\ F_{Xi} \\ \vdots \end{Bmatrix} \qquad \boldsymbol{F}_Y = \begin{Bmatrix} \vdots \\ F_{Yi} \\ \vdots \end{Bmatrix} \qquad \boldsymbol{F}_Z = \begin{Bmatrix} \vdots \\ F_{Zi} \\ \vdots \end{Bmatrix}$$

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\begin{array}{ccc:ccc}
\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z}
\end{array}\right] \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{Bmatrix} \mathrm{d}V$$

External Force vector
for all nodes




Cauchy stress
(Real stress)
Kirchhoff stress
1st Piola-Kirchhoff stress
(Norminal stress)
2nd Piola-Kirchhoff stress
(Pseudo stress)
Stress vector
at each integral node

$$
\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\begin{array}{ccc|ccc}
\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z}
\end{array}\right] \begin{Bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ s_{12} \\ s_{23} \\ s_{31} \end{Bmatrix} \mathrm{d}V
$$

B matrix

at each integral node

Linear B matrix

The same matrix as
an **infinitesimal deformation problem**

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\begin{array}{ccc:ccc}
\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z}
\end{array}\right] \begin{Bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ \hdashline s_{12} \\ s_{23} \\ s_{31} \end{Bmatrix} \mathrm{d}V$$

B matrix

at each integral node

Linear B matrix

The same matrix as
infinitesimal deformation problem

$+$

Non-linear B matrix

Only used for
large deformation problem

$$
\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\begin{array}{ccc:ccc}
\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z}
\end{array} \right] \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{Bmatrix} \mathrm{d}V
$$

B matrix

at each integral node

Finally

we can summarize it into

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\quad \mathbf{B} \quad \right] \left\{ \boldsymbol{s} \right\} \mathrm{d}V$$

To solve
Failure & **Breakage**
problem

$$=$$

To solve

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \begin{bmatrix} & \mathbf{B} & \end{bmatrix} \begin{Bmatrix} \boldsymbol{s} \end{Bmatrix} \mathrm{d}V$$

It is solvable
when **constitutive model** is substituted

$$\boxed{\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \begin{bmatrix} & \mathbf{B} & \end{bmatrix} \begin{Bmatrix} \boldsymbol{s} \end{Bmatrix} \mathrm{d}V} = \int_B \begin{bmatrix} & \mathbf{B} & \end{bmatrix} \begin{bmatrix} & \mathbf{D} & \end{bmatrix} \begin{Bmatrix} \boldsymbol{E} \end{Bmatrix} \mathrm{d}V$$

$$\boxed{\begin{Bmatrix} \boldsymbol{s} \end{Bmatrix} = \begin{bmatrix} & \mathbf{D} & \end{bmatrix} \begin{Bmatrix} \boldsymbol{E} \end{Bmatrix}}$$

It is solvable
when substituting **strain-stress relation**

$$\boxed{\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \begin{bmatrix} \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{E} \end{Bmatrix} \mathrm{d}V} = \int_B \begin{bmatrix} \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{B}^{\mathrm{T}} \end{bmatrix} \mathrm{d}V \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{Bmatrix}$$

$$\boxed{\begin{Bmatrix} \boldsymbol{E} \end{Bmatrix} = \begin{bmatrix} \mathbf{B}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{Bmatrix}}$$

To solve

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\quad \mathbf{B} \quad \right] \left[\quad \mathbf{D} \quad \right] \left[\quad \mathbf{B}^{\mathrm{T}} \quad \right] \mathrm{d}V \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{Bmatrix}$$

Or

rewrite the matrix multiplication into the form

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \begin{bmatrix} & \mathbf{K} & \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{Bmatrix}$$

Seems easy?

$$\begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} = \begin{bmatrix} & \mathbf{K} & \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix}$$

But・・・

Where does this equation come from?

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\quad \mathbf{B} \quad \right] \left\{ \boldsymbol{s} \right\} \mathrm{d}V$$

Where does this equation come from?

Expand **B matrix**

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \left[\begin{array}{ccc|ccc}
\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\
\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z}
\end{array}\right] \left\{\begin{array}{c} s_{11} \\ s_{22} \\ s_{33} \\ \hline s_{12} \\ s_{23} \\ s_{31} \end{array}\right\} \mathrm{d}V$$

The starting point is

Equation of Equilibrium

$$\int_S \boldsymbol{t} \, \mathrm{d}a + \int_B \boldsymbol{b} \, \mathrm{d}v = \boldsymbol{0}$$

And

Deformation Gradient Tensor

$$\mathrm{d}\boldsymbol{x} = \mathbf{F}\mathrm{d}\boldsymbol{X}$$

Equation of Equilibrium

$$\int_S \boldsymbol{t}\,\mathrm{d}a + \int_B \boldsymbol{b}\,\mathrm{d}v = \boldsymbol{0}$$

×

Deformation Gradient Tensor

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\mathrm{d}\boldsymbol{X}$$

Traction （surface force） Body force

$$\int_S \boldsymbol{t}\, da + \int_B \boldsymbol{b}\, dv = \boldsymbol{0}$$

Zero vector

Traction （surface force）

Body force

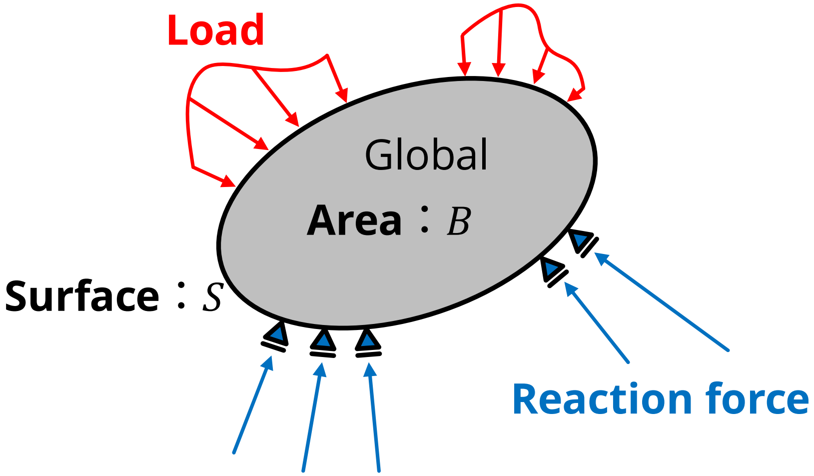
$$\int_S \boldsymbol{t}\, da + \int_B \boldsymbol{b}\, dv = \mathbf{0}$$

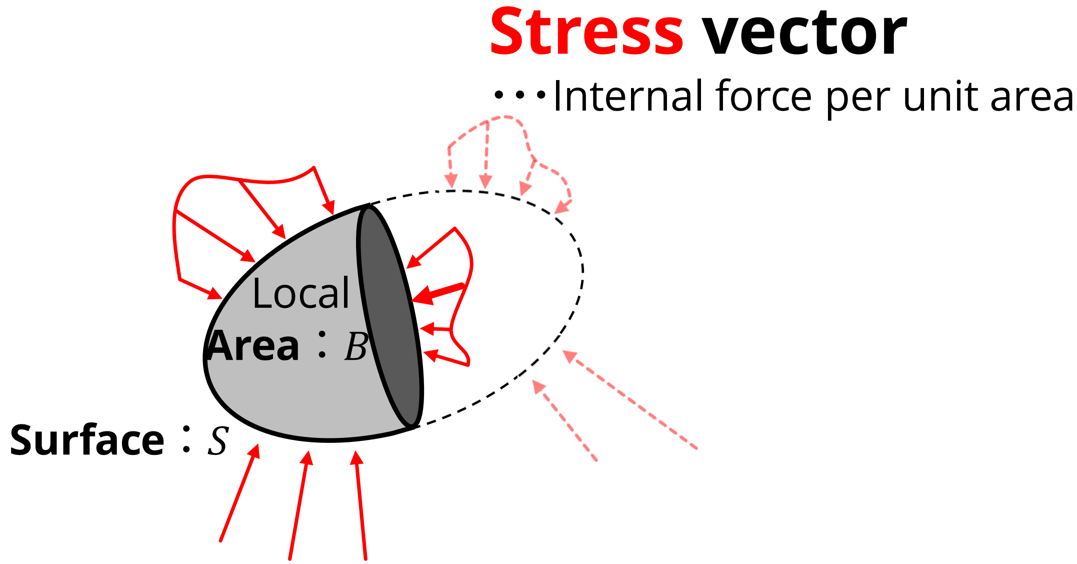
Zero vector

Traction (surface force)

$$\int_S \boldsymbol{t}\, da + \int_B \boldsymbol{b}\, dv = \boldsymbol{0}$$

Body force　Zero vector

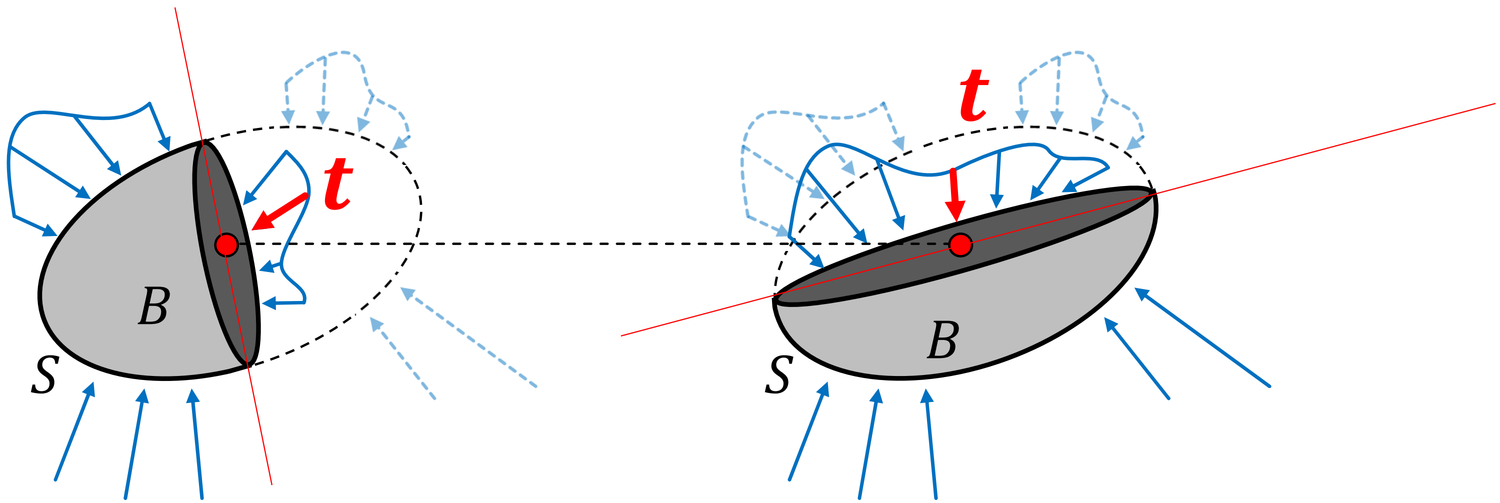




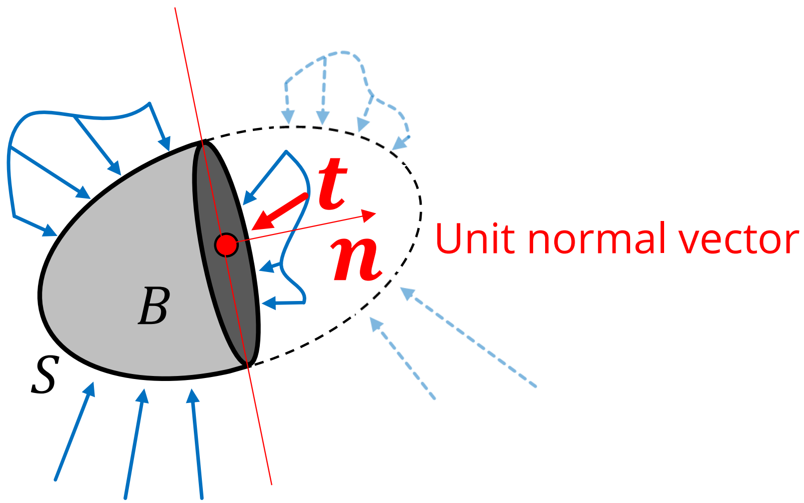
Forces acting on the whole system・・・
External force

Forces acting locally・・・
Internal force

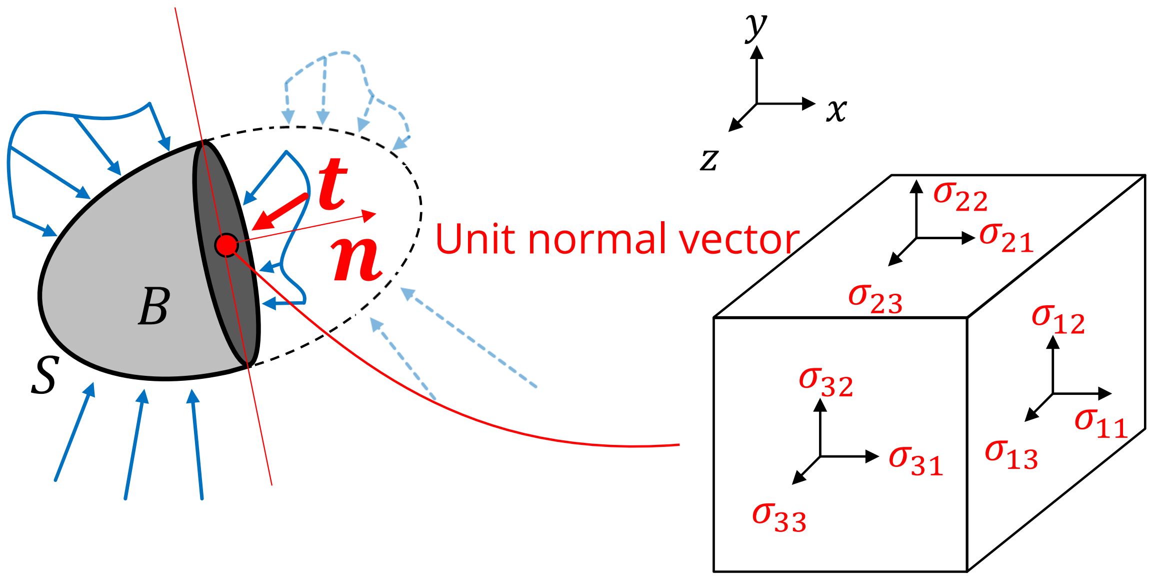
Stress Vector depends on the direction of cross-section



Cross-section is defined by its unit normal vector

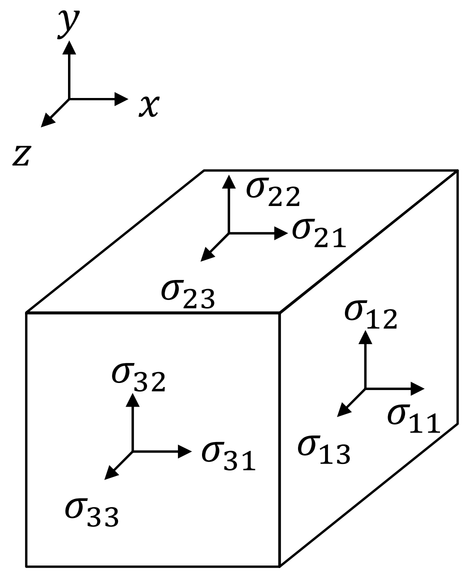


Stress Tensor can be applied to **all directions**



Stress Vector is derived from **Stress Tensor** and **cross-section**

$$\boxed{\boldsymbol{t} = f(\sigma, \boldsymbol{n})}$$

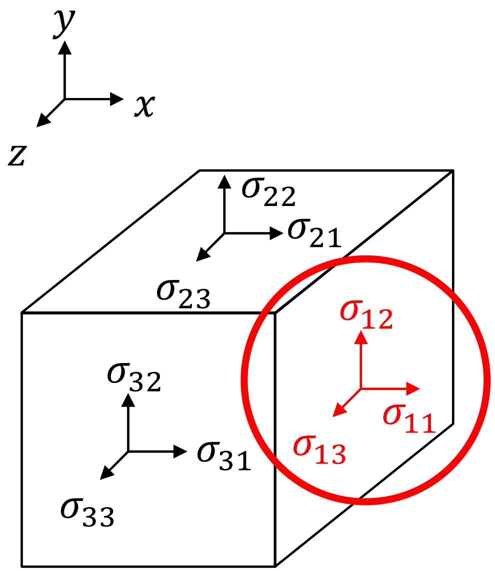


Cross-section x

$$\boldsymbol{t} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix}$$

$$\boldsymbol{n} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\boxed{\boldsymbol{t} = f(\sigma, \boldsymbol{n})}$$

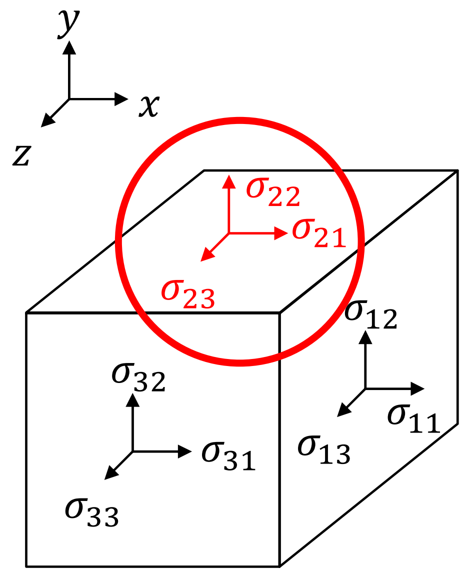


Cross-section x Cross-section y

$$\boldsymbol{t} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} \qquad \boldsymbol{t} = \begin{Bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix}$$

$$\boldsymbol{n} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad \boldsymbol{n} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$\boxed{\boldsymbol{t} = f(\sigma, \boldsymbol{n})}$$

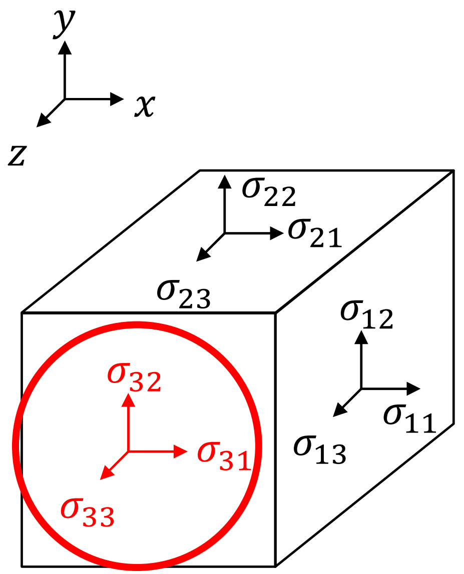


Cross-section x Cross-section y Cross-section z

$$\boldsymbol{t} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} \qquad \boldsymbol{t} = \begin{Bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix} \qquad \boldsymbol{t} = \begin{Bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix}$$

$$\boldsymbol{n} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad \boldsymbol{n} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \qquad \boldsymbol{n} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

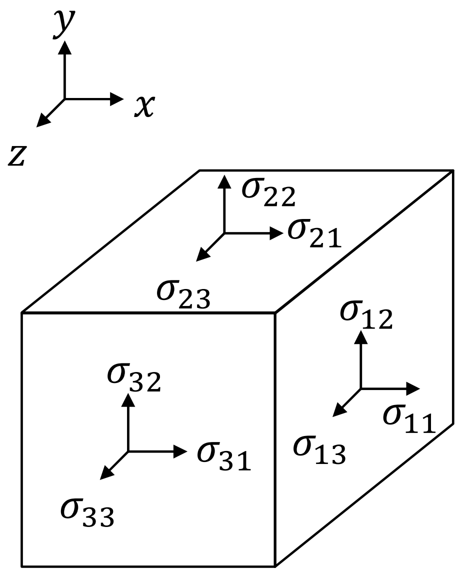
$$\boxed{\boldsymbol{t} = f(\sigma, \boldsymbol{n})}$$



Cross-section x Cross-section y Cross-section z Arbitrary cross-section

$$\boldsymbol{t} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} \quad \boldsymbol{t} = \begin{Bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix} \quad \boldsymbol{t} = \begin{Bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix} \quad \boldsymbol{t} = n_1 \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} + n_2 \begin{Bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix} + n_3 \begin{Bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix}$$

$$\boldsymbol{n} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \boldsymbol{n} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad \boldsymbol{n} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad \boldsymbol{n} = \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \quad |\boldsymbol{n}| = \sqrt{n_1^2 + n_2^2 + n_3^2} = 1$$



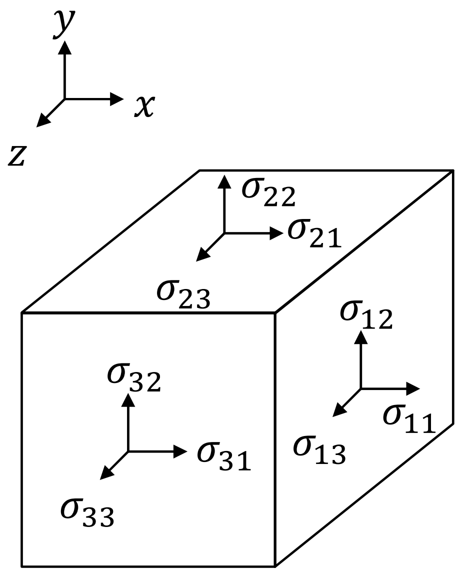
$$\boxed{\boldsymbol{t} = f(\sigma, \boldsymbol{n})}$$

Cross-section x Cross-section y Cross-section z Arbitrary cross-section

$$\boldsymbol{t}=\begin{Bmatrix}\sigma_{11}\\ \sigma_{12}\\ \sigma_{13}\end{Bmatrix}\qquad \boldsymbol{t}=\begin{Bmatrix}\sigma_{21}\\ \sigma_{22}\\ \sigma_{23}\end{Bmatrix}\qquad \boldsymbol{t}=\begin{Bmatrix}\sigma_{31}\\ \sigma_{32}\\ \sigma_{33}\end{Bmatrix}\qquad \boldsymbol{t}=n_1\begin{Bmatrix}\sigma_{11}\\ \sigma_{12}\\ \sigma_{13}\end{Bmatrix}+n_2\begin{Bmatrix}\sigma_{21}\\ \sigma_{22}\\ \sigma_{23}\end{Bmatrix}+n_3\begin{Bmatrix}\sigma_{31}\\ \sigma_{32}\\ \sigma_{33}\end{Bmatrix}$$

$$\boldsymbol{n}=\begin{Bmatrix}1\\ 0\\ 0\end{Bmatrix}\qquad \boldsymbol{n}=\begin{Bmatrix}0\\ 1\\ 0\end{Bmatrix}\qquad \boldsymbol{n}=\begin{Bmatrix}0\\ 0\\ 1\end{Bmatrix}\qquad \boldsymbol{n}=\begin{Bmatrix}n_1\\ n_2\\ n_3\end{Bmatrix}\qquad |\boldsymbol{n}|=\sqrt{n_1^2+n_2^2+n_3^2}=1$$

$$\boxed{\boldsymbol{t}=f(\sigma,\boldsymbol{n})}$$



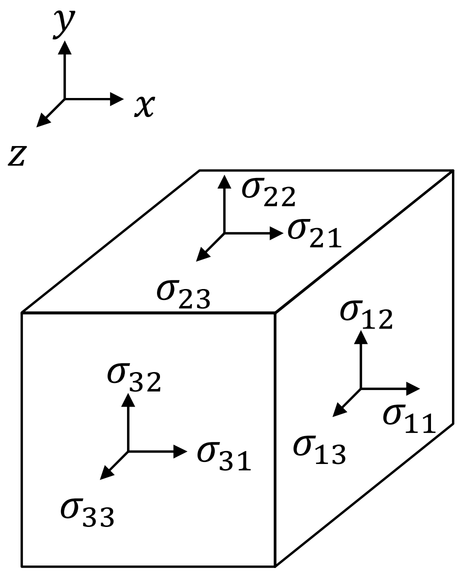
Cross-section x Cross-section y Cross-section z Arbitrary cross-section

$$\boldsymbol{t} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} \qquad \boldsymbol{t} = \begin{Bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix} \qquad \boldsymbol{t} = \begin{Bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix} \qquad \boldsymbol{t} = n_1 \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} + n_2 \begin{Bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix} + n_3 \begin{Bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix}$$

$$\boldsymbol{n} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad \boldsymbol{n} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \qquad \boldsymbol{n} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \qquad \boldsymbol{n} = \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

$$|\boldsymbol{n}| = \sqrt{n_1^2 + n_2^2 + n_3^2} = 1$$

$$\boxed{\boldsymbol{t} = f(\sigma, \boldsymbol{n})}$$

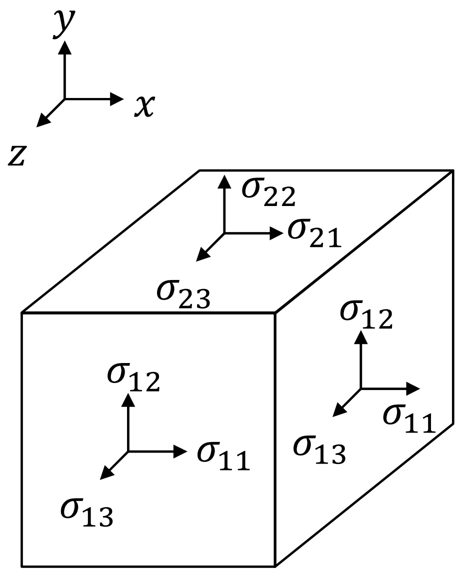


Cross-section x Cross-section y Cross-section z Arbitrary cross-section

$$\boldsymbol{t} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} \qquad \boldsymbol{t} = \begin{Bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix} \qquad \boldsymbol{t} = \begin{Bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix} \qquad \boxed{\boldsymbol{t} = n_1 \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} + n_2 \begin{Bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix} + n_3 \begin{Bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix}}$$

$$\boldsymbol{n} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad \boldsymbol{n} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \qquad \boldsymbol{n} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \qquad \boldsymbol{n} = \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

$$\boxed{\begin{aligned} \boldsymbol{t} &= f(\sigma, \boldsymbol{n}) \\ &= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} \end{aligned}}$$



$$\boldsymbol{t} = n_1 \begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{Bmatrix} + n_2 \begin{Bmatrix} \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \end{Bmatrix} + n_3 \begin{Bmatrix} \sigma_{31} \\ \sigma_{32} \\ \sigma_{33} \end{Bmatrix}$$

$$\Downarrow$$

$$\boldsymbol{t} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix}$$

Matrix form
of stress tensor

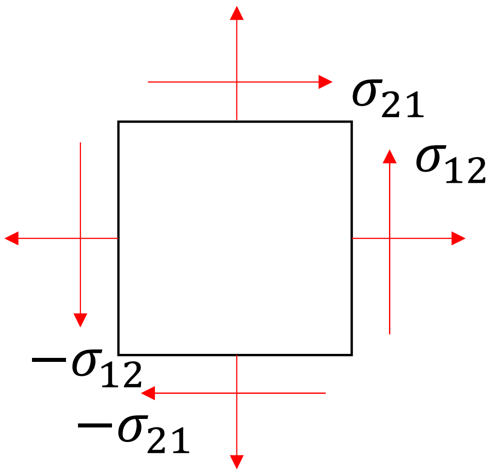
Traction

$$\boldsymbol{t} = [\sigma]^{\mathrm{T}}\{\boldsymbol{n}\}$$

Matrix form of stress tensor

Unit normal vector of cross-section

Matrix form of **Stress Tensor** is **symmetric**



$$\sigma_{ij} = \sigma_{ji}$$

Then **transpose** can be
cancelled out

$$\boldsymbol{t} = [\sigma]\{\boldsymbol{n}\}$$

Substitute $\boxed{\boldsymbol{t} = [\sigma]^{\mathrm{T}}\{\boldsymbol{n}\}}$ **into** $\boxed{\int_S \boldsymbol{t}\, da + \int_B \boldsymbol{b}\, dv = \mathbf{0}}$

$$\int_S [\sigma]^{\mathrm{T}}\boldsymbol{n}\, da + \int_B \boldsymbol{b}\, dv = \mathbf{0}$$

Divergence Theorem

$$\int_S [\quad] \cdot \boldsymbol{n}\, da = \int_B [\quad] \cdot \boldsymbol{\nabla}\, dv$$

$$\begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{Bmatrix}$$

Divergence Theorem

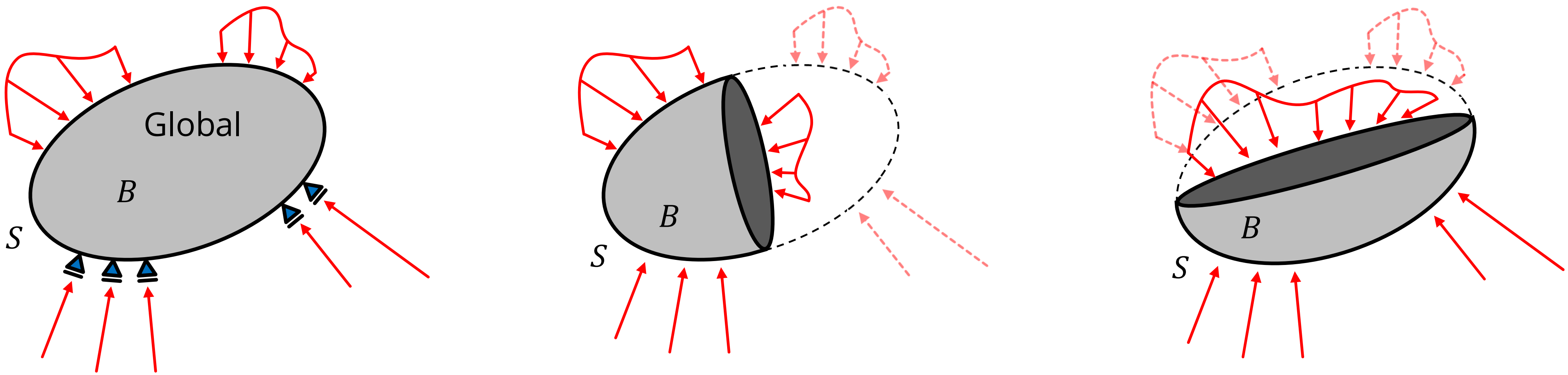
$$\int_S [\sigma]^{\mathrm{T}} \boldsymbol{n}\, da = \int_B \begin{Bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} \end{Bmatrix} dv$$

$$= \int_B \left\{ \frac{\partial \sigma_{ji}}{\partial x_j} \right\} dv$$

j: dummy index

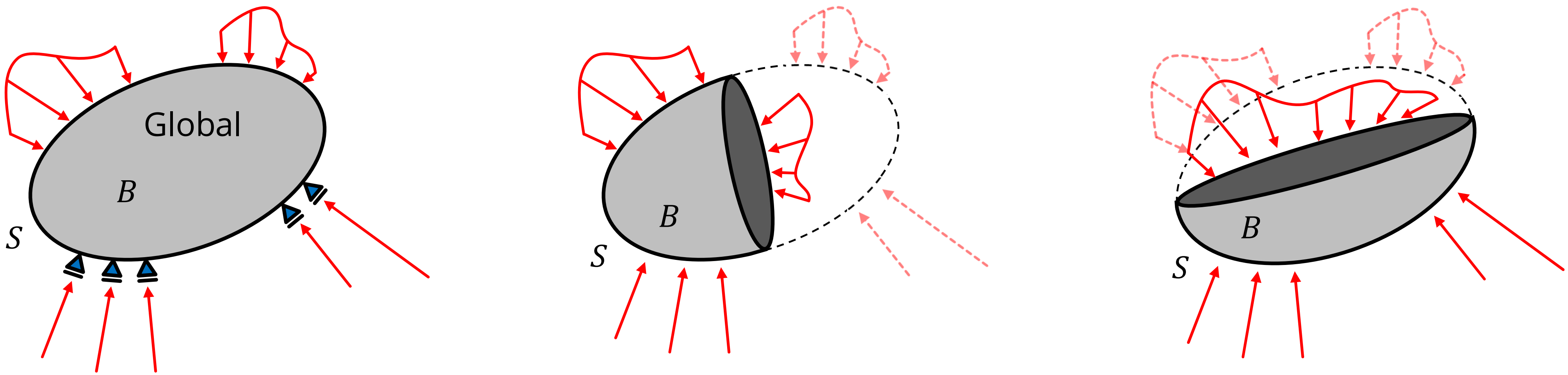
Equation of Equilibrium

$$\int_B \left\{ \frac{\partial \sigma_{ji}}{\partial x_j} \right\} + \boldsymbol{b} \, dv = \mathbf{0}$$



Equation of Equilibrium

$$\left\{\frac{\partial \sigma_{ji}}{\partial x_j}\right\} + \boldsymbol{b} = \mathbf{0}$$



Equation of Equilibrium

$$\frac{\partial \sigma_{ji}}{\partial x_j} + b_i = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \sigma_{ji}}{\partial x_j} = 0}$$

$b_i = 0$

Body force is often ignored

Solve **Differential Equation Numerically**
by **Finite Element Method**

Equation of Equilibrium

External loads
Boundary condition

$$\boxed{\frac{\partial \sigma_{ji}}{\partial x_j} = 0}$$

$$\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix}$$

deformation

Substitute numerical solutions

External loads
Boundary condition

$$\boxed{\frac{\partial \sigma_{ji}}{\partial x_j} = r_i}$$

Numerical solutions

$$\begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{Bmatrix}$$

substitute

Get residual equation

Find the **numerical solutions** which **minimizes** the **residual** = Find the **numerical solutions** where the **average** of **weighted residual** vanishes

Weight vector

$$\int_B \boldsymbol{\omega} \cdot \boldsymbol{r} \, \mathrm{d}v = 0$$

Inner Product

Residual Vector

The whole system

Substitute **Equation of Equilibrium** into **weighted residual equation**

$$\int_B \left\{\omega_i\right\} \cdot \left\{\frac{\partial \sigma_{ji}}{\partial x_j}\right\} \mathrm{d}v = 0$$

Then

$$\int_B \omega_i \frac{\partial \sigma_{ji}}{\partial x_j} \mathrm{d}v$$

is included in

$$\int_B \frac{\partial \omega_i \sigma_{ji}}{\partial x_j} \mathrm{d}v = \int_B \omega_i \frac{\partial \sigma_{ji}}{\partial x_j} \mathrm{d}v + \int_B \frac{\partial \omega_i}{\partial x_j} \sigma_{ji} \mathrm{d}v$$

$$\int_B \omega_i \frac{\partial \sigma_{ji}}{\partial x_j} \mathrm{d}v = 0$$

is included in

By divergence theorem

$$\boxed{\int_S \omega_i \sigma_{ji} n_j \mathrm{d}a} = \int_B \omega_i \frac{\partial \sigma_{ji}}{\partial x_j} \mathrm{d}v + \int_B \frac{\partial \omega_i}{\partial x_j} \sigma_{ji} \mathrm{d}v$$

Equation of Equilibrium

$$\int_S \omega_i \sigma_{ji} n_j \mathrm{d}a = \int_B \frac{\partial \omega_i}{\partial x_j} \sigma_{ji} \mathrm{d}v$$

$$\int_S \{\boldsymbol{\omega}\} \cdot \boxed{\left(\left[\sigma \right]^{\mathrm{T}} \{\boldsymbol{n}\} \right)} \mathrm{d}a = \int_B \left[\frac{\partial \omega_i}{\partial x_j} \right] : \left[\sigma_{ji} \right] \mathrm{d}v$$

$$\boxed{\left(\left[\sigma \right]^{\mathrm{T}} \{\boldsymbol{n}\} \right)} \;\; || \;\; \{\boldsymbol{t}\}$$

(inner product of tensor)

Summation of all components

$$\frac{\partial \omega_1}{\partial x_1}\sigma_{11} + \frac{\partial \omega_1}{\partial x_2}\sigma_{21} + \frac{\partial \omega_1}{\partial x_3}\sigma_{31} + \frac{\partial \omega_2}{\partial x_1}\sigma_{12} + \cdots$$

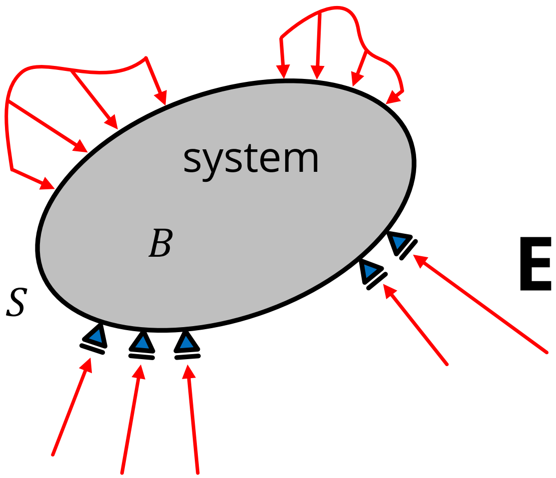
Weighted Residual Equation

is defined for the whole system

$$\int_S \{\boldsymbol{\omega}\} \cdot \{\boldsymbol{t}\} \, \mathrm{d}a = \int_B \left[\frac{\partial \omega_i}{\partial x_j}\right] : \left[\sigma_{ji}\right] \mathrm{d}v$$

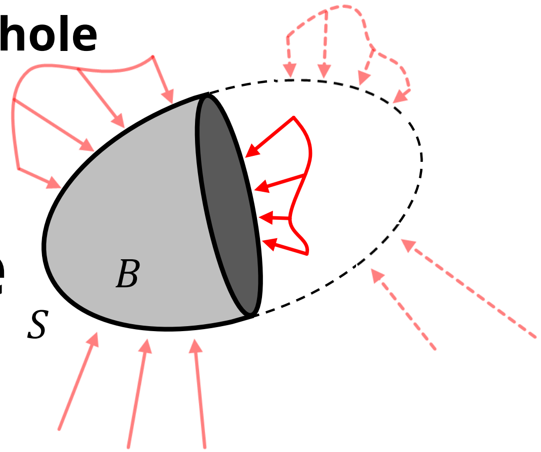
Traction
on the surface
of system

Stress
of the whole
system



External force

Internal force



Traction on the surface of system

External force

- **Concentrated loads**
- **Distributed loads**

Deformation and **Weight** are replaced by **approximate solutions**

$$\int_S \{\boldsymbol{\omega}\} \cdot \{\boldsymbol{t}\}\, \mathrm{d}a = \int_B \left[\frac{\partial \omega_i}{\partial x_j}\right] : \left[\sigma_{ji}\right] \mathrm{d}v$$

Before substitution,
Make it into a weak form

substitute

approximate solutions

$$\begin{array}{ccccccc} & & & n & n & n & & \\ 3 & \{\boldsymbol{\omega}\} & = & \multicolumn{3}{c}{\left[\begin{array}{ccc} \boldsymbol{N}^{\mathrm{T}} & & \\ & \boldsymbol{N}^{\mathrm{T}} & \\ & & \boldsymbol{N}^{\mathrm{T}} \end{array}\right]} & \begin{Bmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \\ \boldsymbol{\omega}_3 \end{Bmatrix} & 3n \end{array}$$

$$\begin{array}{ccccc} 3 & \{\boldsymbol{u}\} & = & \left[\begin{array}{ccc} \boldsymbol{N}^{\mathrm{T}} & & \\ & \boldsymbol{N}^{\mathrm{T}} & \\ & & \boldsymbol{N}^{\mathrm{T}} \end{array}\right] & \begin{Bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \boldsymbol{u}_3 \end{Bmatrix} & 3n \end{array}$$

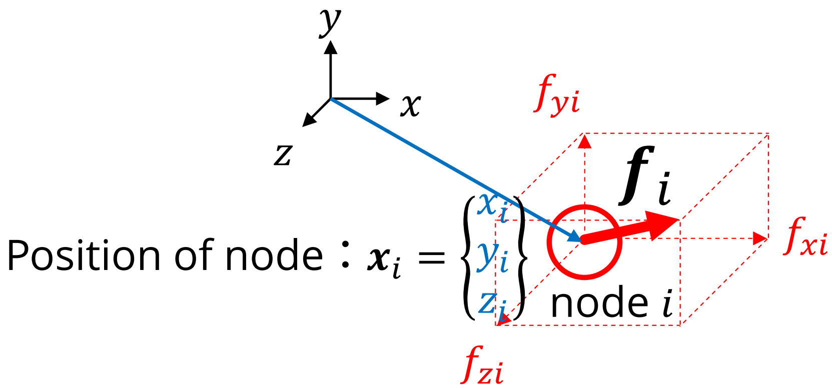
$\boldsymbol{u}$ is derived from σ_{ji}

Finally, the left-hand side of **finite element equation** becomes the **load vector** for each degree of freedom

$$\int_S \{\boldsymbol{\omega}\}\cdot\{\boldsymbol{t}\}\,\mathrm{d}a = \int_B \left[\frac{\partial \omega_i}{\partial x_j}\right] : \left[\sigma_{ji}\right]\mathrm{d}v$$

$$\boldsymbol{f}_x = \begin{Bmatrix} \vdots \\ f_{xi} \\ \vdots \end{Bmatrix} \qquad \boldsymbol{f}_y = \begin{Bmatrix} \vdots \\ f_{yi} \\ \vdots \end{Bmatrix} \qquad \boldsymbol{f}_z = \begin{Bmatrix} \vdots \\ f_{zi} \\ \vdots \end{Bmatrix}$$

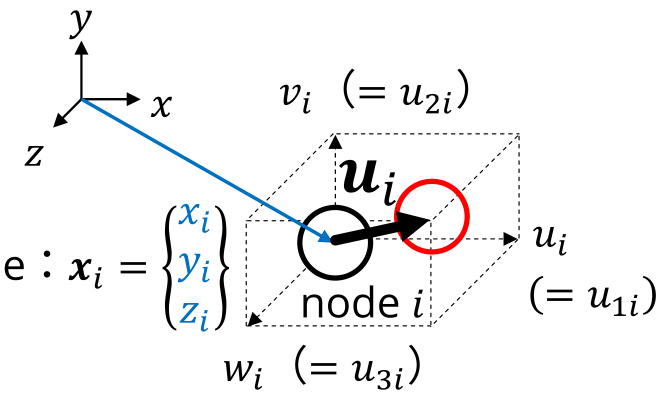
$$\begin{bmatrix} \{\boldsymbol{f}_x\} \\ \{\boldsymbol{f}_y\} \\ \{\boldsymbol{f}_z\} \end{bmatrix}$$



And the right-hand side of **finite element equation** becomes the **stiffness vector** and **deformation vector**

$$\int_S \{\boldsymbol{\omega}\} \cdot \{\boldsymbol{t}\}\, \mathrm{d}a = \int_B \left[\frac{\partial \omega_i}{\partial x_j}\right] : \left[\sigma_{ji}\right] \mathrm{d}v$$

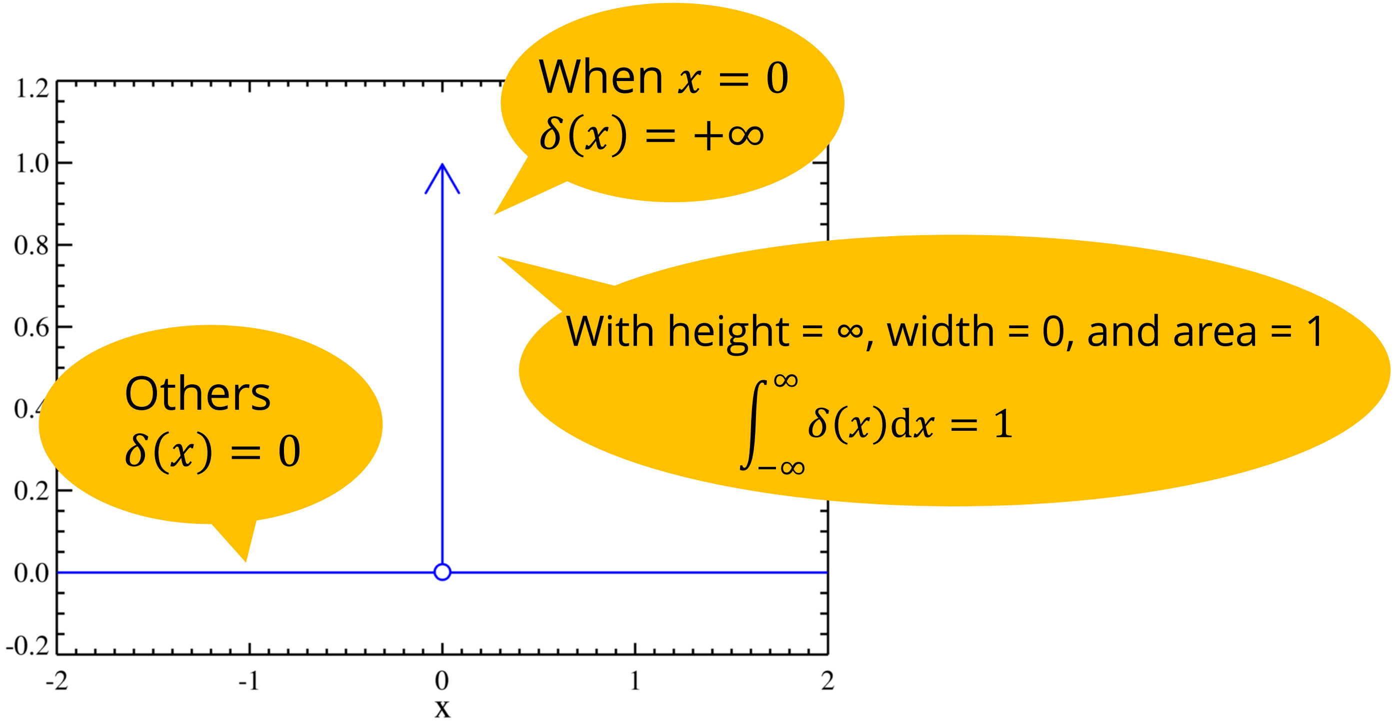
Position of node : $\boldsymbol{x}_i = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}$



$$\begin{bmatrix} & & \\ & \mathbf{K} & \\ & & \end{bmatrix} \begin{bmatrix} \{\boldsymbol{u}_1\} \\ \{\boldsymbol{u}_2\} \\ \{\boldsymbol{u}_3\} \end{bmatrix}$$

$$\boldsymbol{u}_1 = \begin{Bmatrix} \vdots \\ u_{1i} \\ \vdots \end{Bmatrix} \qquad \boldsymbol{u}_2 = \begin{Bmatrix} \vdots \\ u_{2i} \\ \vdots \end{Bmatrix} \qquad \boldsymbol{u}_3 = \begin{Bmatrix} \vdots \\ u_{3i} \\ \vdots \end{Bmatrix}$$

Concentrated load is a **Delta function**



Dirac delta function

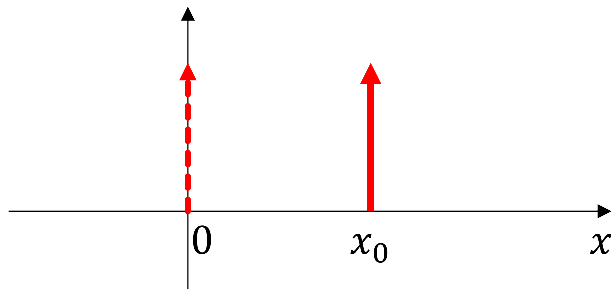
How to use Delta function

Any function

$$\int_{-\infty}^{\infty} \boxed{\delta(x - x_0)} F(x)\mathrm{d}x \; = \; F(x_0)$$

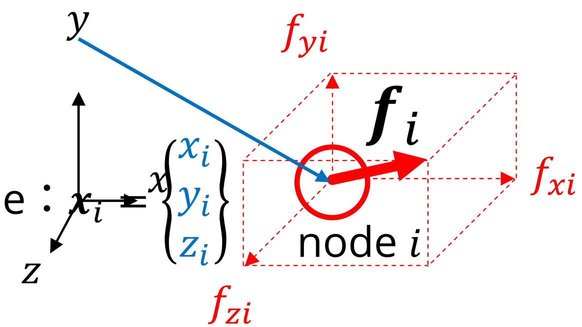
Returns the value only at $x = x_0$

Impulse at $x = x_0$



Traction on the surface of system
is the **concentrated** **loads** on all nodes

Position of node：$\boldsymbol{x}_i = \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix}$



$$\begin{Bmatrix} \boldsymbol{t} \end{Bmatrix} = \begin{Bmatrix} \cdots + \delta(\boldsymbol{x} - \boldsymbol{x}_i) f_{xi} + \cdots \\ \cdots + \delta(\boldsymbol{x} - \boldsymbol{x}_i) f_{yi} + \cdots \\ \cdots + \delta(\boldsymbol{x} - \boldsymbol{x}_i) f_{zi} + \cdots \end{Bmatrix}$$

Traction

Extend Delta function
into three dimensions

Most terms are zeros
※　$\boldsymbol{x} = \boldsymbol{x_i}$で、$\{\boldsymbol{t}\} = \{\boldsymbol{f}_i\}$

By interpolation
approximate solutions
of **weight**
$\omega_i = \boldsymbol{\omega}_i^T \boldsymbol{N}$

$N_j(x, y, z)$ equal to **1** at node j
0 at all other nodes

$$\int_S \overset{3\times 1}{\{\boldsymbol{\omega}\}} \cdot \{\boldsymbol{t}\} \, \mathrm{d}a = \overset{n\times 1}{[\boldsymbol{\omega}_1^T \quad \boldsymbol{\omega}_2^T \quad \boldsymbol{\omega}_3^T]} \boxed{\int_S \begin{bmatrix} \{\boldsymbol{N}\} & & \\ & \{\boldsymbol{N}\} & \\ & & \{\boldsymbol{N}\} \end{bmatrix} \{\boldsymbol{t}\} \, \mathrm{d}a}$$

Traction on the surface of system

Weights for each **degree of freedom**

$\boldsymbol{t}$ **is** **The impulse of Delta function at each node** $\times$ **Concentrated loads of each direction at each node**

3 (number of dimensions)

$$\int_S \{\boldsymbol{\omega}\} \cdot \{\boldsymbol{t}\}\,\mathrm{d}a = \underbrace{[\boldsymbol{\omega}_1^T \quad \boldsymbol{\omega}_2^T \quad \boldsymbol{\omega}_3^T]}_{\text{Degree of freedom：3n}} \boxed{\int_S \begin{bmatrix} \boxed{\{\boldsymbol{N}\}\quad\quad} & & \\ & \{\boldsymbol{N}\} & \\ & & \{\boldsymbol{N}\} \end{bmatrix} \boxed{\{\boldsymbol{t}\}}\,\mathrm{d}a}$$

$$\left\{ \begin{matrix} \sum_k^n \delta(\boldsymbol{x} - \boldsymbol{x}_k) \cdot f_{xk} \\ \sum_k^n \delta(\boldsymbol{x} - \boldsymbol{x}_k) \cdot f_{yk} \\ \sum_k^n \delta(\boldsymbol{x} - \boldsymbol{x}_k) \cdot f_{zk} \end{matrix} \right\}$$

j changes from 1 to n

$$= [\boldsymbol{\omega}_1^T \quad \boldsymbol{\omega}_2^T \quad \boldsymbol{\omega}_3^T] \left\{ \begin{matrix} \vdots \\ \int_S N_j(x, y, z) \cdot \left(\sum_k^n \delta(\boldsymbol{x} - \boldsymbol{x}_k) \cdot f_{xk} \right) \mathrm{d}a \\ \vdots \\ \\ \vdots \\ \\ \vdots \end{matrix} \right\}$$

Degree of freedom：3n

3 (number of dimensions)

$$\int_S \{\boldsymbol{\omega}\}\cdot\{\boldsymbol{t}\}\,\mathrm{d}a = \underbrace{[\boldsymbol{\omega}_1^T \quad \boldsymbol{\omega}_2^T \quad \boldsymbol{\omega}_3^T]}_{\text{Degree of freedom: 3n}} \boxed{\int_S \begin{bmatrix} \{\boldsymbol{N}\} & & \\ & \boxed{\{\boldsymbol{N}\}} & \\ & & \{\boldsymbol{N}\} \end{bmatrix} \boxed{\{\boldsymbol{t}\}}\,\mathrm{d}a}$$

$$= [\boldsymbol{\omega}_1^T \quad \boldsymbol{\omega}_2^T \quad \boldsymbol{\omega}_3^T] \begin{Bmatrix} \vdots \\ \int_S N_j(x,y,z)\cdot\left(\sum_k^n \delta(\boldsymbol{x}-\boldsymbol{x}_k)\cdot f_{xk}\right)\mathrm{d}a \\ \vdots \\ \int_S N_j(x,y,z)\cdot\left(\sum_k^n \delta(\boldsymbol{x}-\boldsymbol{x}_k)\cdot f_{yk}\right)\mathrm{d}a \\ \vdots \\ \vdots \end{Bmatrix}$$

j changes from 1 to n

Degree of freedom：3n

3 (number of dimensions)

$$\int_S \{\boldsymbol{\omega}\}\cdot\{\boldsymbol{t}\}\,\mathrm{d}a = \underbrace{[\boldsymbol{\omega}_1^T \quad \boldsymbol{\omega}_2^T \quad \boldsymbol{\omega}_3^T]}_{\text{Degree of freedom：3n}} \boxed{\int_S \begin{bmatrix} \{\boldsymbol{N}\} & & \\ & \{\boldsymbol{N}\} & \\ & & \{\boldsymbol{N}\} \end{bmatrix} \boxed{\{\boldsymbol{t}\}}\,\mathrm{d}a}$$

j changes from 1 to n

$$= [\boldsymbol{\omega}_1^T \quad \boldsymbol{\omega}_2^T \quad \boldsymbol{\omega}_3^T] \begin{Bmatrix} \vdots \\ \int_S N_j(x,y,z)\cdot\left(\sum_k^n \delta(\boldsymbol{x}-\boldsymbol{x}_k)\cdot f_{xk}\right)\mathrm{d}a \\ \vdots \\ \int_S N_j(x,y,z)\cdot\left(\sum_k^n \delta(\boldsymbol{x}-\boldsymbol{x}_k)\cdot f_{yk}\right)\mathrm{d}a \\ \vdots \\ \int_S N_j(x,y,z)\cdot\left(\sum_k^n \delta(\boldsymbol{x}-\boldsymbol{x}_k)\cdot f_{zk}\right)\mathrm{d}a \\ \vdots \end{Bmatrix}$$

Degree of freedom：3n

3 (number of dimensions)

$$\int_S \{\boldsymbol{\omega}\}\cdot\{\boldsymbol{t}\}\,\mathrm{d}a = \underbrace{[\boldsymbol{\omega}_1^T \quad \boldsymbol{\omega}_2^T \quad \boldsymbol{\omega}_3^T]}_{\text{Degree of freedom : 3n}} \boxed{\int_S \begin{bmatrix} \{\boldsymbol{N}\} & & \\ & \{\boldsymbol{N}\} & \\ & & \{\boldsymbol{N}\} \end{bmatrix} \{\boldsymbol{t}\}\,\mathrm{d}a}$$

j changes from 1 to n

$$= [\boldsymbol{\omega}_1^T \quad \boldsymbol{\omega}_2^T \quad \boldsymbol{\omega}_3^T] \begin{Bmatrix} \vdots \\ \int_S N_j(x,y,z)\cdot\left(\sum_k^n \delta(\boldsymbol{x}-\boldsymbol{x}_k)\cdot f_{xk}\right)\mathrm{d}a \\ \vdots \\ \int_S N_j(x,y,z)\cdot\left(\sum_k^n \delta(\boldsymbol{x}-\boldsymbol{x}_k)\cdot f_{yk}\right)\mathrm{d}a \\ \vdots \\ \int_S N_j(x,y,z)\cdot\left(\sum_k^n \delta(\boldsymbol{x}-\boldsymbol{x}_k)\cdot f_{zk}\right)\mathrm{d}a \\ \vdots \end{Bmatrix}$$

Degree of freedom : 3n

1 at node (x_j, y_j, z_j)
0 for others

0 for j = k

$$\int_S N_j(x,y,z)\cdot\left(\sum_k^n \delta(\boldsymbol{x}-\boldsymbol{x}_k)\cdot f_{xk}\right)\mathrm{d}a$$

$$=\int_S \left(\sum_k^n \delta(\boldsymbol{x}-\boldsymbol{x}_k)\cdot N_j(x,y,z) f_{xk}\right)\mathrm{d}a$$

$$=\left(\sum_k^n \boxed{N_j(x_k,y_k,z_k)} f_{xk}\right)$$

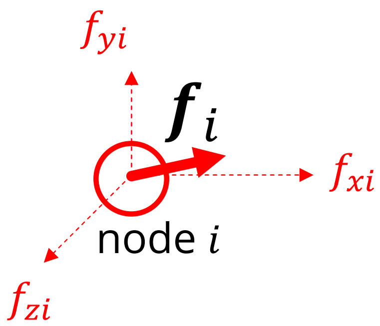
$$= f_{xj}$$

1 at node (x_j, y_j, z_j)

0 for others

Only load vector **at each node** remains

$$\int_S \{\boldsymbol{w}\} \cdot \{\boldsymbol{t}\} \, \mathrm{d}a = [\boldsymbol{W}_1^T \quad \boldsymbol{W}_2^T \quad \boldsymbol{W}_3^T] \begin{bmatrix} \{\boldsymbol{f}_{\boldsymbol{x}}\} \\ \{\boldsymbol{f}_{\boldsymbol{y}}\} \\ \{\boldsymbol{f}_{\boldsymbol{z}}\} \end{bmatrix}$$



Next deal with the right-hand side

$$\begin{bmatrix} \boldsymbol{W}_1^T & \boldsymbol{W}_2^T & \boldsymbol{W}_3^T \end{bmatrix} \begin{bmatrix} \{\boldsymbol{f}_x\} \\ \{\boldsymbol{f}_y\} \\ \{\boldsymbol{f}_z\} \end{bmatrix} = \boxed{\int_B \left[\frac{\partial \omega_i}{\partial x_j}\right] : [\sigma_{ji}] \mathrm{d}v}$$

$$= \begin{bmatrix} \boldsymbol{W}_1^T & \boldsymbol{W}_2^T & \boldsymbol{W}_3^T \end{bmatrix} \begin{bmatrix} & & \\ & \mathbf{K} & \\ & & \end{bmatrix} \begin{bmatrix} \{\boldsymbol{U}\} \\ \{\boldsymbol{V}\} \\ \{\boldsymbol{W}\} \end{bmatrix}$$

$$\Downarrow$$

Traction on surface of system **(external force)**

$$\begin{bmatrix} \{\boldsymbol{f}_x\} \\ \{\boldsymbol{f}_y\} \\ \{\boldsymbol{f}_z\} \end{bmatrix} = \begin{bmatrix} & & \\ & \mathbf{K} & \\ & & \end{bmatrix} \begin{bmatrix} \{\boldsymbol{U}\} \\ \{\boldsymbol{V}\} \\ \{\boldsymbol{W}\} \end{bmatrix}$$

Weight function

i refers to x、y、z

dimension＝3 (directions)

Number of node：n

$$\begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial z} \\ \vdots & \vdots & \vdots \\ \frac{\partial N_n}{\partial x} & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial z} \end{bmatrix} \begin{Bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{Bmatrix}$$

$$\int_B \left[\frac{\partial w_i}{\partial x_j}\right] : [\sigma_{ji}]\mathrm{d}v = \int_B \left[\frac{\partial (\boldsymbol{W}_i \cdot \boldsymbol{N})}{\partial x_j}\right] : [\sigma_{ji}]\mathrm{d}v$$

$$= \boldsymbol{W}_1^{\mathrm{T}} \int_B \left[\frac{\partial \boldsymbol{N}}{\partial x_j}\right]\{\sigma_{j1}\}\mathrm{d}v + \boldsymbol{W}_2^{\mathrm{T}} \int_B \left[\frac{\partial \boldsymbol{N}}{\partial x_j}\right]\{\sigma_{j2}\}\mathrm{d}v + \boldsymbol{W}_3^{\mathrm{T}} \int_B \left[\frac{\partial \boldsymbol{N}}{\partial x_j}\right]\{\sigma_{j3}\}\mathrm{d}v$$

Tensor inner product

$$[A_{ij}]:[B_{ij}] \\ = A_{11}B_{11} + A_{12}B_{12} + A_{13}B_{13} \\ + A_{21}B_{21} + A_{22}B_{22} + A_{23}B_{23} \\ + A_{31}B_{31} + A_{32}B_{32} + A_{33}B_{33}$$

$$= [\boldsymbol{W}_1^{\mathrm{T}} \quad \boldsymbol{W}_2^{\mathrm{T}} \quad \boldsymbol{W}_3^{\mathrm{T}}] \int_B \begin{bmatrix} \left[\frac{\partial \boldsymbol{N}}{\partial x_j}\right] & & \\ & \left[\frac{\partial \boldsymbol{N}}{\partial x_j}\right] & \\ & & \left[\frac{\partial \boldsymbol{N}}{\partial x_j}\right] \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix} \mathrm{d}v$$

Reorganize the order

$$\int_B \left[\frac{\partial w_i}{\partial x_j}\right] : [\sigma_{ji}]\mathrm{d}v = \begin{bmatrix} \boldsymbol{W}_1^{\mathrm{T}} & \boldsymbol{W}_2^{\mathrm{T}} & \boldsymbol{W}_3^{\mathrm{T}} \end{bmatrix} \int_B \begin{bmatrix} \left[\frac{\partial \boldsymbol{N}}{\partial x_j}\right] & & \\ & \left[\frac{\partial \boldsymbol{N}}{\partial x_j}\right] & \\ & & \left[\frac{\partial \boldsymbol{N}}{\partial x_j}\right] \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{Bmatrix} \mathrm{d}v$$

$$= \begin{bmatrix} \boldsymbol{W}_1^{\mathrm{T}} & \boldsymbol{W}_2^{\mathrm{T}} & \boldsymbol{W}_3^{\mathrm{T}} \end{bmatrix} \int_B \boxed{\begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \mathrm{d}v$$

B matrix

$$\begin{bmatrix} \boldsymbol{W}_1^T & \boldsymbol{W}_2^T & \boldsymbol{W}_3^T \end{bmatrix} \begin{bmatrix} \{\boldsymbol{f}_x\} \\ \{\boldsymbol{f}_y\} \\ \{\boldsymbol{f}_z\} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W}_1^{\mathrm{T}} & \boldsymbol{W}_2^{\mathrm{T}} & \boldsymbol{W}_3^{\mathrm{T}} \end{bmatrix} \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \mathrm{d}v$$

$$\begin{bmatrix} \{\boldsymbol{f}_x\} \\ \{\boldsymbol{f}_y\} \\ \{\boldsymbol{f}_z\} \end{bmatrix} = \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \mathrm{d}v$$

Right hand side is
Numerical integral at each node

$$\begin{bmatrix} \{\boldsymbol{f}_x\} \\ \{\boldsymbol{f}_y\} \\ \{\boldsymbol{f}_z\} \end{bmatrix} = \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \mathrm{d}v$$

Transform stress to deformation

Assume **Hooke's Law** for **strain-stress relation**

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \underbrace{\boxed{\begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{bmatrix}}}_{\mathbf{D}} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{Bmatrix}$$

Transform stress to strain

Apply
Infinitesimal
Deformation
Theorem
to
Strain-Deformation
Relation

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & & \\ & \frac{\partial}{\partial y} & \\ & & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \\ & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

Replace **deformation** by **approximate solutions**

$$
\begin{aligned}
u &= \boldsymbol{u} \cdot \boldsymbol{N} & v &= \boldsymbol{v} \cdot \boldsymbol{N} & w &= \boldsymbol{w} \cdot \boldsymbol{N} \\
&= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{u} & &= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{v} & &= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{w}
\end{aligned}
$$

$$
\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{Bmatrix}
=
\begin{bmatrix}
\frac{\partial}{\partial x} & & \\
& \frac{\partial}{\partial y} & \\
& & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \\
& \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial z} & & \frac{\partial}{\partial x}
\end{bmatrix}
\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}
=
\underbrace{\begin{bmatrix}
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial x} & & \\
& \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial y} & \\
& & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial z} \\
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial y} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial x} & \\
& \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial z} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial y} \\
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial z} & & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial x}
\end{bmatrix}}_{\mathbf{B}^{\mathrm{T}}}
\begin{bmatrix} \{\boldsymbol{u}\} \\ \{\boldsymbol{v}\} \\ \{\boldsymbol{w}\} \end{bmatrix}
$$

$$\{\boldsymbol{f}\} = \int_B [\mathbf{B}]\{\boldsymbol{\sigma}\}\,\mathrm{d}v$$

$$= \int_B [\mathbf{B}][\mathbf{D}]\{\boldsymbol{\varepsilon}\}\,\mathrm{d}v$$

$$= \int_B [\mathbf{B}][\mathbf{D}][\mathbf{B}]^{\mathrm{T}}\{\boldsymbol{u}\}\mathrm{d}v$$

$$= \int_B [\mathbf{B}][\mathbf{D}][\mathbf{B}]^{\mathrm{T}}\mathrm{d}v\,\{\boldsymbol{u}\}$$

$$\begin{bmatrix} \{\boldsymbol{f}_x\} \\ \{\boldsymbol{f}_y\} \\ \{\boldsymbol{f}_z\} \end{bmatrix} = \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \mathrm{d}v$$

$$= \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{Bmatrix} \mathrm{d}v$$

$$= \left(\int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{bmatrix} \begin{bmatrix} \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial x} & & \\ & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial y} & \\ & & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial z} \\ \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial y} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial x} & \\ & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial z} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial y} \\ \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial z} & & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial x} \end{bmatrix} \mathrm{d}v \right) \begin{bmatrix} \{\boldsymbol{u}\} \\ \{\boldsymbol{v}\} \\ \{\boldsymbol{w}\} \end{bmatrix}$$

If **stiffness matrix** $[\mathbf{K}]$ can be calculated
Then we got **numerical solutions**

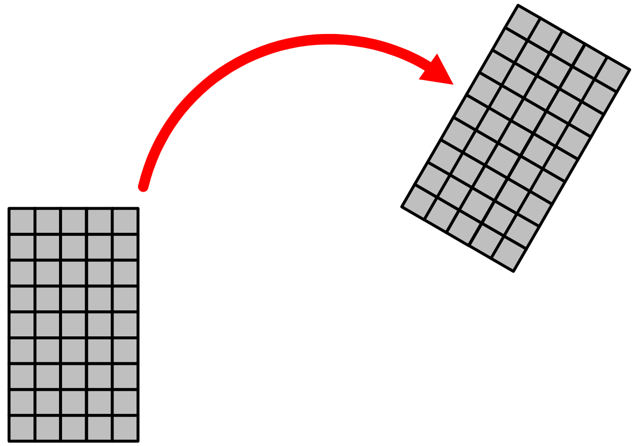
$$\{\boldsymbol{f}\} = \boxed{\int_B [\mathbf{B}][\mathbf{D}][\mathbf{B}]^{\mathrm{T}} \mathrm{d}v} \{\boldsymbol{u}\} \Rightarrow \boxed{\{\boldsymbol{f}\} = [\mathbf{K}]\{\boldsymbol{u}\}}$$

After applying boundary condition
Solve the group of equations
(matrix inverse problem)

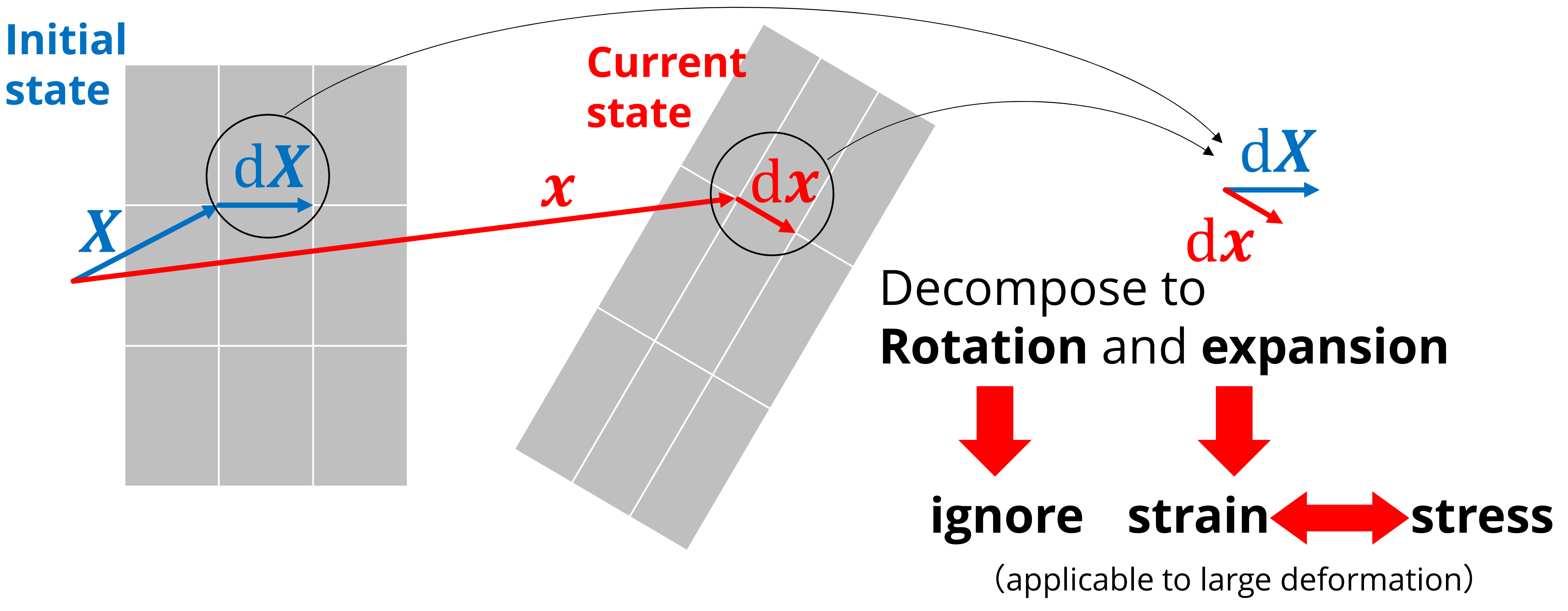
Infinitesimal deformation is the **basic assumption**

Infinitesimal strain is not **zero**
Because of **rotation** of **rigid body**

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & & \\ & \dfrac{\partial}{\partial y} & \\ & & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & \\ & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} & & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$



Redefine strain
to cancel out the influence of
transition and **rotation** of rigid body,
so that strain is **infinitesimal**

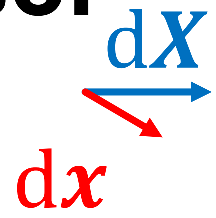


Initial state
Current state
$\mathrm{d}\boldsymbol{X}$
$\boldsymbol{X}$
$\boldsymbol{x}$
$\mathrm{d}\boldsymbol{x}$
$\mathrm{d}\boldsymbol{X}$
$\mathrm{d}\boldsymbol{x}$
Decompose to
Rotation and **expansion**
ignore
strain
stress
(applicable to large deformation)

Components of

Deformation Gradient Tensor

$$\begin{Bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \begin{Bmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \\ \mathrm{d}X_3 \end{Bmatrix}$$



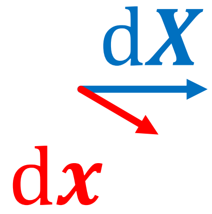
Total derivative

$$dx = \frac{\partial x}{\partial X} dX + \frac{\partial x}{\partial Y} dY + \frac{\partial x}{\partial Z} dZ$$

$$\begin{Bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{Bmatrix} = \underbrace{\boxed{\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}}}_{\mathbf{F}} \begin{Bmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \\ \mathrm{d}X_3 \end{Bmatrix}$$

Deformation Gradient Tensor

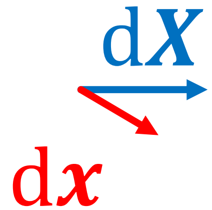
$$\{\mathrm{d}x_i\} = \left[\frac{\partial x_i}{\partial X_j}\right]\{\mathrm{d}X_j\}$$



j: dummy index

Deformation Gradient Tensor

$$\mathrm{d}\boldsymbol{x} = \mathbf{F}\,\mathrm{d}\boldsymbol{X}$$



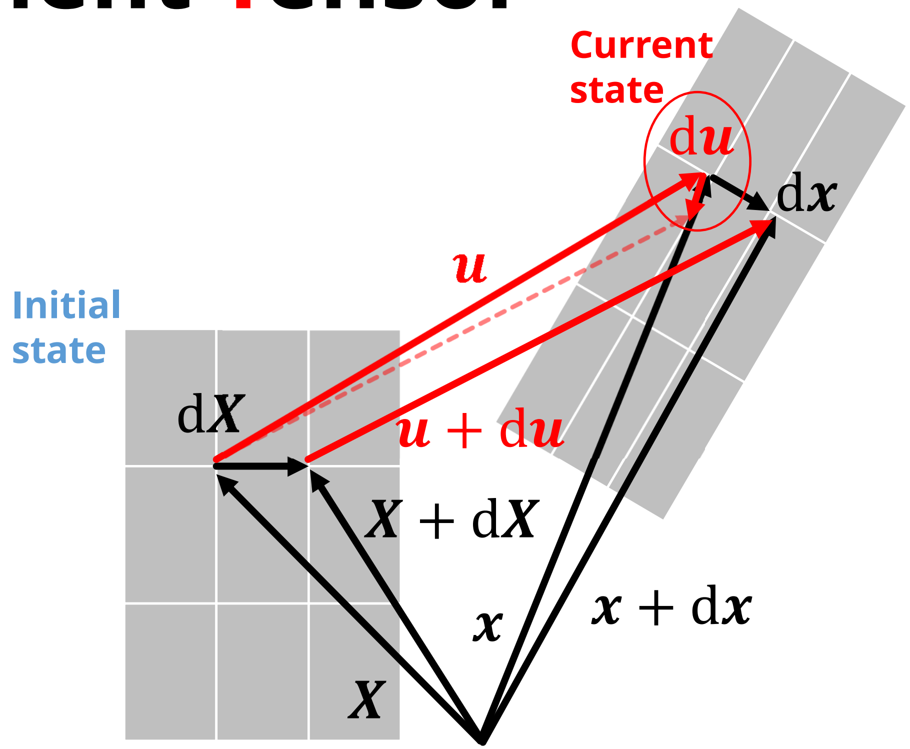
Deformation Vector
and
Deformation Gradient Tensor

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$$

$$\begin{aligned}\boldsymbol{u} + \mathrm{d}\boldsymbol{u} &= (\boldsymbol{x} + \mathrm{d}\boldsymbol{x}) - (\boldsymbol{X} + \mathrm{d}\boldsymbol{X}) \\ &= \boldsymbol{x} - \boldsymbol{X} + \mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X} \\ &= \boldsymbol{u} + \mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X}\end{aligned}$$

$$\begin{aligned}\mathrm{d}\boldsymbol{u} &= \mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X} \\ &= \mathbf{F}\mathrm{d}\boldsymbol{X} - \mathrm{d}\boldsymbol{X} = (\mathbf{F} - \mathbf{I})\mathrm{d}\boldsymbol{X}\end{aligned}$$

$\mathrm{d}\boldsymbol{x} = \mathbf{F}\mathrm{d}\boldsymbol{X}$

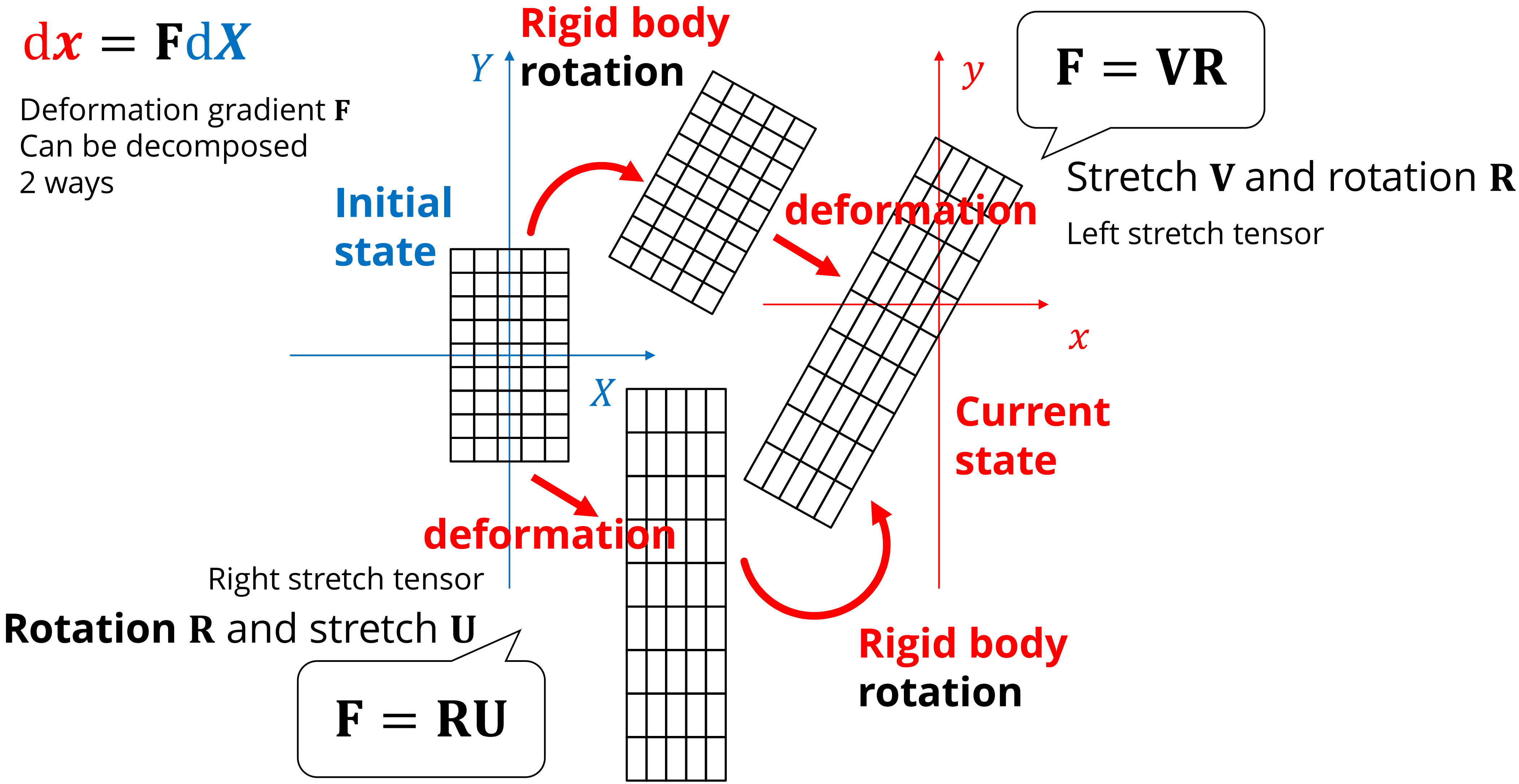


$$\boxed{\mathrm{d}\boldsymbol{u} = (\mathbf{F} - \mathbf{I})\mathrm{d}\boldsymbol{X}} \Longleftrightarrow \boxed{\mathrm{d}\boldsymbol{u} = \left[\frac{\partial u_i}{\partial X_j}\right]\mathrm{d}\boldsymbol{X}}$$

$$\mathbf{F} - \mathbf{I} = \left[\frac{\partial u_i}{\partial X_j}\right]$$

$$\boxed{\mathbf{F} = \left[\frac{\partial u_i}{\partial X_j}\right] + \mathbf{I}}$$

$$F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$



$\mathrm{d}\boldsymbol{x} = \mathbf{F}\mathrm{d}\boldsymbol{X}$
Deformation gradient $\mathbf{F}$
Can be decomposed
2 ways
Initial state
Rigid body rotation
deformation
Current state
$\mathbf{F} = \mathbf{V}\mathbf{R}$
Stretch $\mathbf{V}$ and rotation $\mathbf{R}$
Left stretch tensor
deformation
Rigid body rotation
Right stretch tensor
Rotation $\mathbf{R}$ and stretch $\mathbf{U}$
$\mathbf{F} = \mathbf{R}\mathbf{U}$
X
Y
x
y

Stress is derived from **stretch tensor**

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

Rotation tensor is desired to be **cancelled out**

Symmetric matrix

$\mathbf{U} = \mathbf{U}^{\mathrm{T}}$

$\mathbf{V} = \mathbf{V}^{\mathrm{T}}$

Orthogonal matrix

$\mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$

$$\boxed{\begin{aligned} \mathbf{F} &= \mathbf{R}\mathbf{U} \\ \mathbf{F}^{\mathrm{T}}\mathbf{F} &= \mathbf{U}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{U} \\ &= \mathbf{U}^{\mathrm{T}}\mathbf{U} \\ &= \mathbf{U}^{2} \end{aligned}}$$

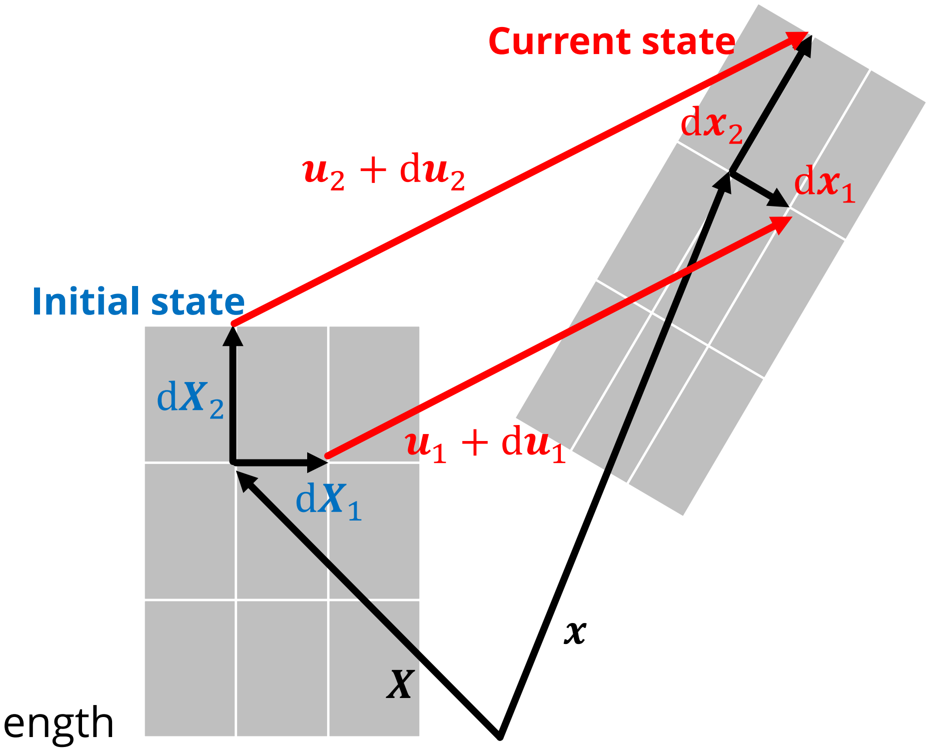
$$\begin{aligned} \mathbf{F} &= \mathbf{V}\mathbf{R} \\ \mathbf{F}\mathbf{F}^{\mathrm{T}} &= \mathbf{V}\mathbf{R}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} \\ &= \mathbf{V}\mathbf{V}^{\mathrm{T}} \\ &= \mathbf{V}^{2} \end{aligned}$$

Green-Lagrange strain

$$\begin{aligned}\mathrm{d}\boldsymbol{x}_1 \cdot \mathrm{d}\boldsymbol{x}_2 &= (\mathbf{F}\mathrm{d}\boldsymbol{X}_1)\cdot(\mathbf{F}\mathrm{d}\boldsymbol{X}_2)\\ &= \mathrm{d}\boldsymbol{X}_1^{\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathrm{d}\boldsymbol{X}_2\\ &= \mathrm{d}\boldsymbol{X}_1^{\mathrm{T}}\mathbf{U}^2\mathrm{d}\boldsymbol{X}_2\end{aligned}$$

$$\begin{aligned}\mathbf{U}^2 = \mathbf{F}^{\mathrm{T}}\mathbf{F} &= \left(\left[\frac{\partial u_j}{\partial X_i}\right] + \mathbf{I}\right)\left(\left[\frac{\partial u_i}{\partial X_j}\right] + \mathbf{I}\right)\\ &= \left(\underbrace{\boxed{\left[\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right] + \left[\frac{\partial u_j}{\partial X_i}\right] + \left[\frac{\partial u_i}{\partial X_j}\right]}}_{2E_{ij}} + \mathbf{I}\right)\end{aligned}$$

Original length



Total derivative

$$dx = \frac{\partial x}{\partial X} dX + \frac{\partial x}{\partial Y} dY + \frac{\partial x}{\partial Z} dZ$$

$$\begin{Bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}}_{\mathbf{F}} \begin{Bmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \\ \mathrm{d}X_3 \end{Bmatrix}$$

Partial derivative

$$\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial \omega}{\partial Y}\frac{\partial Y}{\partial x} + \frac{\partial \omega}{\partial Z}\frac{\partial Z}{\partial x}$$

$$\begin{Bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_2}{\partial x_1} & \frac{\partial X_3}{\partial x_1} \\ \frac{\partial X_1}{\partial x_2} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_3}{\partial x_2} \\ \frac{\partial X_1}{\partial x_3} & \frac{\partial X_2}{\partial x_3} & \frac{\partial X_3}{\partial x_3} \end{bmatrix}}_{\mathbf{F}^{-\mathrm{T}}} \begin{Bmatrix} \partial/\partial X_1 \\ \partial/\partial X_2 \\ \partial/\partial X_3 \end{Bmatrix}$$

Deformation gradient tensor

$$\begin{Bmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \begin{Bmatrix} \mathrm{d}X_1 \\ \mathrm{d}X_2 \\ \mathrm{d}X_3 \end{Bmatrix} \qquad \begin{Bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_2}{\partial x_1} & \frac{\partial X_3}{\partial x_1} \\ \frac{\partial X_1}{\partial x_2} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_3}{\partial x_2} \\ \frac{\partial X_1}{\partial x_3} & \frac{\partial X_2}{\partial x_3} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} \begin{Bmatrix} \partial/\partial X_1 \\ \partial/\partial X_2 \\ \partial/\partial X_3 \end{Bmatrix}$$

$$\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_2}{\partial x_1} & \frac{\partial X_3}{\partial x_1} \\ \frac{\partial X_1}{\partial x_2} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_3}{\partial x_2} \\ \frac{\partial X_1}{\partial x_3} & \frac{\partial X_2}{\partial x_3} & \frac{\partial X_3}{\partial x_3} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonal component

$$\frac{\partial x_1}{\partial X_1}\frac{\partial X_1}{\partial x_1} + \frac{\partial x_1}{\partial X_2}\frac{\partial X_2}{\partial x_1} + \frac{\partial x_1}{\partial X_3}\frac{\partial X_3}{\partial x_1}$$

$$= \frac{x_1}{x_1} = 1$$

Non-diagonal component

$$\frac{\partial x_1}{\partial X_1}\frac{\partial X_1}{\partial x_2} + \frac{\partial x_1}{\partial X_2}\frac{\partial X_2}{\partial x_2} + \frac{\partial x_1}{\partial X_3}\frac{\partial X_3}{\partial x_2}$$

$$= \frac{x_1}{x_2} = 0$$

Nanson's equation

Cross production of 2 vectors after coordination transform

Transform $\boldsymbol{x}$ and $\boldsymbol{y}$ to $\boldsymbol{a}$ and $\boldsymbol{b}$ Using $\mathbf{F}$

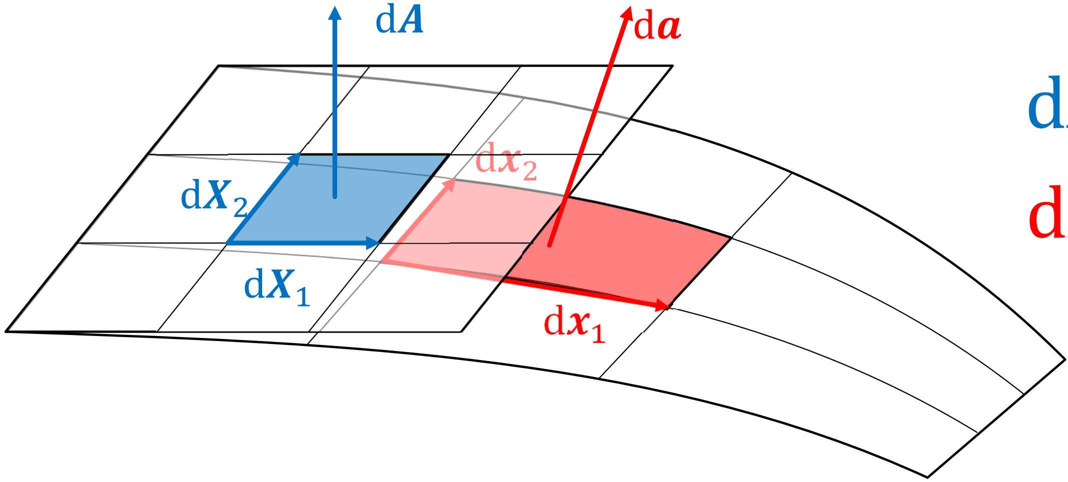
$$\boxed{\boldsymbol{a} \times \boldsymbol{b} = (\mathbf{F}\boldsymbol{x}) \times (\mathbf{F}\boldsymbol{y})}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \times \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{Bmatrix} = \begin{Bmatrix} F_{2j} x_j F_{3l} y_l - F_{3j} x_j F_{2l} y_l \\ F_{3j} x_j F_{1l} y_l - F_{1j} x_j F_{3l} y_l \\ F_{1j} x_j F_{2l} y_l - F_{2j} x_j F_{1l} y_l \end{Bmatrix} = \begin{Bmatrix} F_{2i} F_{3j} \left(x_i y_j - x_j y_i \right) \\ F_{3i} F_{1j} \left(x_i y_j - x_j y_i \right) \\ F_{1i} F_{2j} \left(x_i y_j - x_j y_i \right) \end{Bmatrix}$$

$$\boxed{a_i = F_{ij} x_j \qquad b_k = F_{kl} y_l}$$

$$= \begin{bmatrix} F_{22}F_{33} - F_{23}F_{32} & F_{21}F_{33} - F_{23}F_{31} & F_{22}F_{33} - F_{23}F_{32} \\ F_{32}F_{13} - F_{33}F_{12} & F_{31}F_{13} - F_{33}F_{31} & F_{31}F_{32} - F_{32}F_{31} \\ F_{12}F_{23} - F_{13}F_{22} & F_{11}F_{23} - F_{13}F_{21} & F_{11}F_{22} - F_{12}F_{21} \end{bmatrix} \begin{Bmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{Bmatrix}$$

$$\boxed{= J \mathbf{F}^{-\mathrm{T}} (\boldsymbol{x} \times \boldsymbol{y})}$$

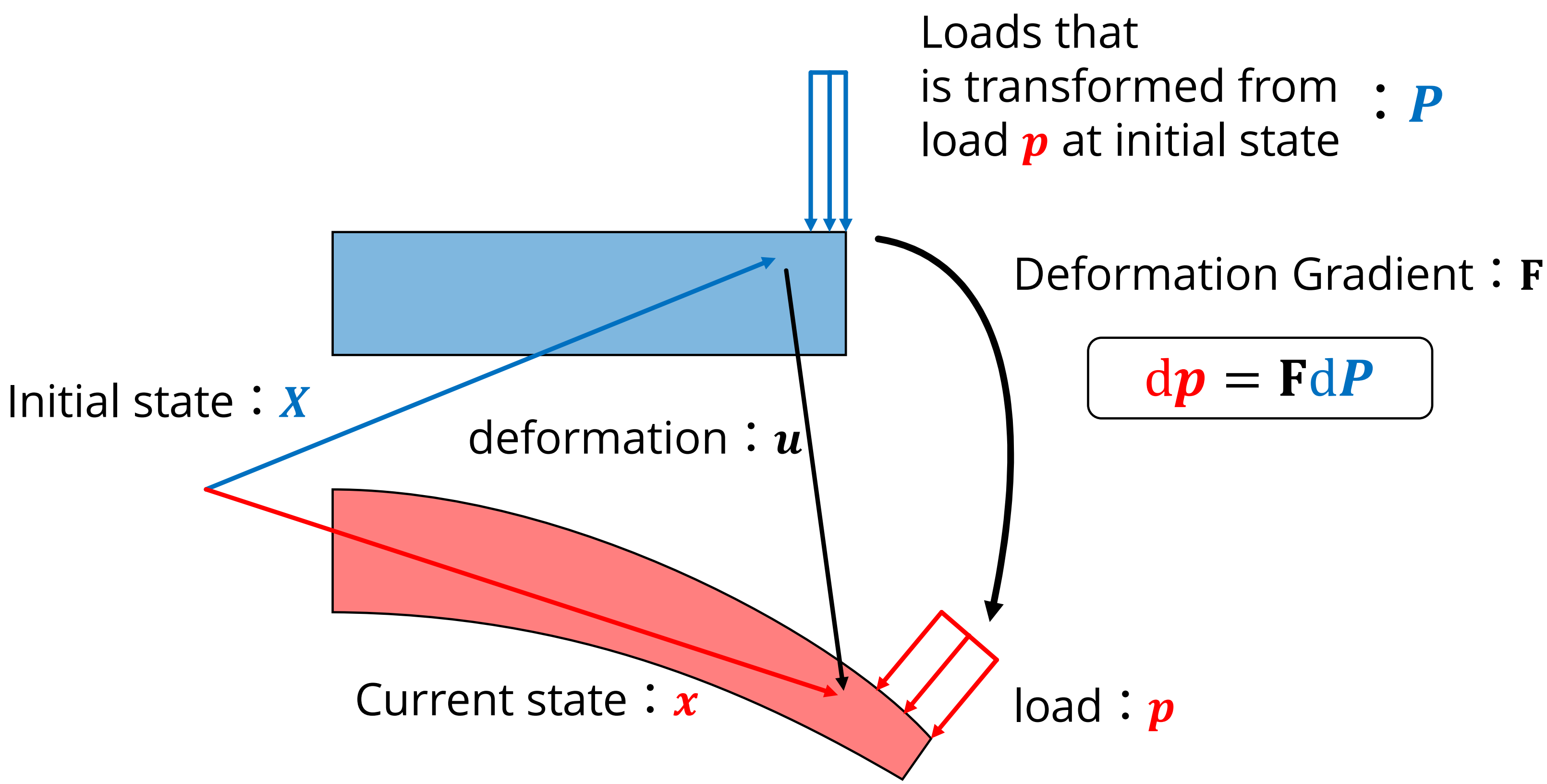


Nanson's equation

$$\mathrm{d}\boldsymbol{A} = \boldsymbol{N}\mathrm{d}A = \mathrm{d}\boldsymbol{X}_1 \times \mathrm{d}\boldsymbol{X}_2$$

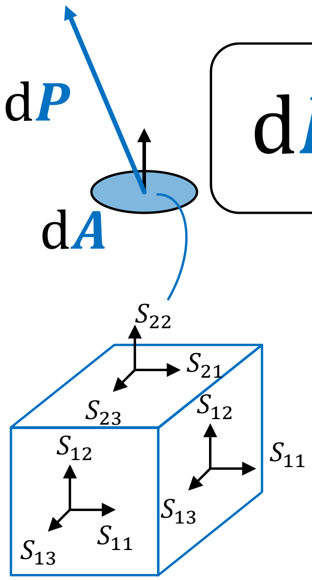
$$\begin{aligned}
\mathrm{d}\boldsymbol{a} = \boldsymbol{n}\mathrm{d}a &= \mathrm{d}\boldsymbol{x}_1 \times \mathrm{d}\boldsymbol{x}_2 \\
&= (\mathbf{F}\mathrm{d}\boldsymbol{X}_1) \times (\mathbf{F}\mathrm{d}\boldsymbol{X}_2) \\
&= J\mathbf{F}^{-\mathrm{T}}(\mathrm{d}\boldsymbol{X}_1 \times \mathrm{d}\boldsymbol{X}_2) \\
&= J\mathbf{F}^{-\mathrm{T}}\mathrm{d}\boldsymbol{A}
\end{aligned}$$

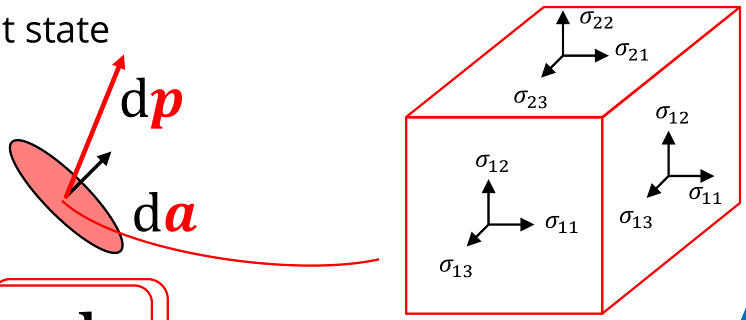
$$\boxed{\boldsymbol{n}\mathrm{d}a = J\mathbf{F}^{-\mathbf{T}}\boldsymbol{N}\mathrm{d}A}$$


Loads that
is transformed from
load p at initial state
: P
Deformation Gradient : F
dp = FdP
Initial state : X
deformation : u
Current state : x
load : p

Virtual force
at initial state

Real force
at current state





$$\mathrm{d}\boldsymbol{P} = \mathbf{F}^{-1}\mathrm{d}\boldsymbol{p}$$
$$= \mathbf{F}^{-1}[\boldsymbol{\sigma}]^{\mathrm{T}}\boldsymbol{n}\mathrm{d}a$$

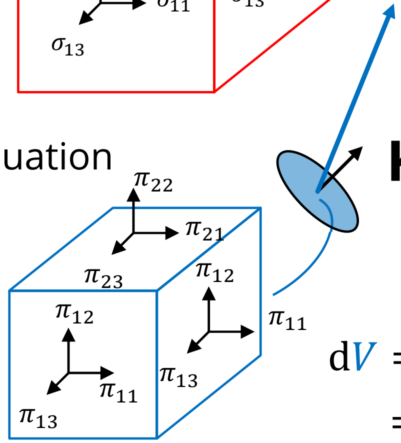
Nanson's equation

$$= \mathbf{F}^{-1}[\boldsymbol{\sigma}]^{\mathrm{T}} J\mathbf{F}^{-\mathrm{T}}\boldsymbol{N}\mathrm{d}A$$
$$= \mathbf{F}^{-1}[\boldsymbol{\pi}]^{\mathrm{T}}\mathbf{F}^{-\mathrm{T}}\boldsymbol{N}\mathrm{d}A$$
$$= \mathbf{F}^{-1}[\boldsymbol{\Pi}]^{\mathrm{T}}\boldsymbol{N}\mathrm{d}A$$
$$= [\mathbf{S}]^{\mathrm{T}}\boldsymbol{N}\mathrm{d}A$$

Cauchy stress $[\boldsymbol{\sigma}]$

$$\mathrm{d}v = (\mathrm{d}\boldsymbol{x}_1 \times \mathrm{d}\boldsymbol{x}_2)\cdot \mathrm{d}\boldsymbol{x}_3$$

Only magnitudes
are the same



Kirchhoff stress

$$[\boldsymbol{\pi}] = J[\boldsymbol{\sigma}]$$

$$\begin{aligned}
\mathrm{d}V &= (\mathrm{d}\boldsymbol{X}_1 \times \mathrm{d}\boldsymbol{X}_2)\cdot \mathrm{d}\boldsymbol{X}_3\\
&= J\mathbf{F}^{-\mathbf{T}}(\mathrm{d}\boldsymbol{x}_1 \times \mathrm{d}\boldsymbol{x}_2)\cdot \mathbf{F}\mathrm{d}\boldsymbol{x}_3\\
&= J(\mathrm{d}\boldsymbol{x}_1 \times \mathrm{d}\boldsymbol{x}_2)^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{F}\mathrm{d}\boldsymbol{x}_3\\
&= J(\mathrm{d}\boldsymbol{x}_1 \times \mathrm{d}\boldsymbol{x}_2)^{\mathrm{T}}\mathrm{d}\boldsymbol{x}_3 = J\mathrm{d}\boldsymbol{v}
\end{aligned}$$

(nominal stress) Return to original
deformation

1st Piola-Kirchhoff stress

$$[\boldsymbol{\Pi}] = \mathbf{F}^{-1}[\boldsymbol{\pi}]$$

(pseudo stress) Apply symmetry

2nd Piola-Kirchhoff stress

$$[\mathbf{S}] = [\boldsymbol{\Pi}]\mathbf{F}^{-\mathrm{T}}$$

2nd Piola-Kirchhoff stress

$$[\mathbf{S}] = J\mathbf{F}^{-1}[\boldsymbol{\sigma}]\mathbf{F}^{-\mathrm{T}}$$

$$[\boldsymbol{\sigma}] = \frac{1}{J}\mathbf{F}[\mathbf{S}]\mathbf{F}^{\mathrm{T}}$$

$$\sigma_{11} = F_{1i}S_{ij}F_{1j} = F_{11}S_{11}F_{11} + F_{11}S_{12}F_{12} + F_{11}S_{13}F_{13} + F_{12}S_{21}F_{11} + F_{12}S_{22}F_{12} + F_{12}S_{23}F_{13} + F_{13}S_{31}F_{11} + F_{13}S_{32}F_{12} + F_{13}S_{33}F_{13}$$

$$\sigma_{22} = F_{2i}S_{ij}F_{2j} = F_{21}S_{11}F_{21} + F_{21}S_{12}F_{22} + F_{21}S_{13}F_{23} + F_{22}S_{21}F_{21} + F_{22}S_{22}F_{22} + F_{22}S_{23}F_{23} + F_{23}S_{31}F_{21} + F_{23}S_{32}F_{22} + F_{23}S_{33}F_{23}$$

$$\sigma_{33} = F_{3i}S_{ij}F_{3j} = F_{31}S_{11}F_{31} + F_{31}S_{12}F_{32} + F_{31}S_{13}F_{33} + F_{32}S_{21}F_{31} + F_{32}S_{22}F_{32} + F_{32}S_{23}F_{33} + F_{33}S_{31}F_{31} + F_{33}S_{32}F_{32} + F_{33}S_{33}F_{33}$$

$$\sigma_{12} = F_{1i}S_{ij}F_{2j} = F_{11}S_{11}F_{21} + F_{11}S_{12}F_{22} + F_{11}S_{13}F_{23} + F_{12}S_{21}F_{21} + F_{12}S_{22}F_{22} + F_{12}S_{23}F_{23} + F_{13}S_{31}F_{21} + F_{13}S_{32}F_{22} + F_{13}S_{33}F_{23}$$

$$\sigma_{23} = F_{2i}S_{ij}F_{3j} = F_{21}S_{11}F_{31} + F_{21}S_{12}F_{32} + F_{21}S_{13}F_{33} + F_{22}S_{21}F_{31} + F_{22}S_{22}F_{32} + F_{22}S_{23}F_{33} + F_{23}S_{31}F_{31} + F_{23}S_{32}F_{32} + F_{23}S_{33}F_{33}$$

$$\sigma_{31} = F_{3i}S_{ij}F_{1j} = F_{31}S_{11}F_{11} + F_{31}S_{12}F_{12} + F_{31}S_{13}F_{13} + F_{32}S_{21}F_{11} + F_{32}S_{22}F_{12} + F_{32}S_{23}F_{13} + F_{33}S_{31}F_{11} + F_{33}S_{32}F_{12} + F_{33}S_{33}F_{13}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\ F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\ F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\ F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22} + F_{12}F_{21} & F_{12}F_{23} + F_{13}F_{22} & F_{11}F_{23} + F_{13}F_{21} \\ F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32} + F_{22}F_{31} & F_{22}F_{33} + F_{23}F_{32} & F_{21}F_{33} + F_{23}F_{31} \\ F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12} + F_{32}F_{11} & F_{32}F_{13} + F_{33}F_{12} & F_{31}F_{13} + F_{33}F_{11} \end{bmatrix} \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{Bmatrix}$$

right hand side of equation of equilibrium of finite element method

$$\int_B \left[\frac{\partial w_i}{\partial x_j}\right] : [\sigma_{ji}]\mathrm{d}v = [\boldsymbol{W}_1^{\mathrm{T}} \quad \boldsymbol{W}_2^{\mathrm{T}} \quad \boldsymbol{W}_3^{\mathrm{T}}] \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \mathrm{d}v$$

substitute

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\ F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\ F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\ F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22} + F_{12}F_{21} & F_{12}F_{23} + F_{13}F_{22} & F_{11}F_{23} + F_{13}F_{21} \\ F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32} + F_{22}F_{31} & F_{22}F_{33} + F_{23}F_{32} & F_{21}F_{33} + F_{23}F_{31} \\ F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12} + F_{32}F_{11} & F_{32}F_{13} + F_{33}F_{12} & F_{31}F_{13} + F_{33}F_{11} \end{bmatrix} \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{Bmatrix}$$

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\
& \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\
& & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$
\boxed{F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}}
$$

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$F_{ij} = \frac{\partial x_i}{\partial X_j}$ and $F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$

=

$\frac{\partial \mathbf{N}}{\partial x}F_{11}F_{11} + \frac{\partial \mathbf{N}}{\partial y}F_{11}F_{21} + \frac{\partial \mathbf{N}}{\partial z}F_{31}F_{11}$					

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

$=$

$\left(\frac{\partial \mathbf{N}}{\partial x}F_{11} + \frac{\partial \mathbf{N}}{\partial y}F_{21} + \frac{\partial \mathbf{N}}{\partial z}F_{31}\right)F_{11}$					

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$F_{ij} = \dfrac{\partial x_i}{\partial X_j}$ and $F_{ij} = \dfrac{\partial u_i}{\partial X_j} + \delta_{ij}$

=

$\left(\frac{\partial \mathbf{N}}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial \mathbf{N}}{\partial y}\frac{\partial y}{\partial X} + \frac{\partial \mathbf{N}}{\partial z}\frac{\partial z}{\partial X}\right)\left(\frac{\partial u}{\partial X} + 1\right)$					

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

$=$

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X} + 1\right)$					

B matrix in current state

Transform matrix from real stress to pseudo stress

$$\begin{bmatrix} \frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\ & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\ & & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x} \end{bmatrix} \begin{bmatrix} F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\ F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\ F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\ F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\ F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\ F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11} \end{bmatrix}$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

$=$

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$				

B matrix in current state

Transform matrix from real stress to pseudo stress

$$\begin{bmatrix} \frac{\partial N}{\partial x} & & & \frac{\partial N}{\partial y} & & \frac{\partial N}{\partial z} \\ & \frac{\partial N}{\partial y} & & \frac{\partial N}{\partial x} & \frac{\partial N}{\partial z} & \\ & & \frac{\partial N}{\partial z} & & \frac{\partial N}{\partial y} & \frac{\partial N}{\partial x} \end{bmatrix} \begin{bmatrix} F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\ F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\ F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\ F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\ F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\ F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11} \end{bmatrix}$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

=

$\frac{\partial N}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial N}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial N}{\partial Z}\frac{\partial u}{\partial Z}$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\
& \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\
& & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

$=$

$\frac{\partial \boldsymbol{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial u}{\partial Z}$			
$\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial v}{\partial X}$					

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
 & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
 & & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

=

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z}$			
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$				

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22} + F_{12}F_{21} & F_{12}F_{23} + F_{13}F_{22} & F_{11}F_{23} + F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32} + F_{22}F_{31} & F_{22}F_{33} + F_{23}F_{32} & F_{21}F_{33} + F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12} + F_{32}F_{11} & F_{32}F_{13} + F_{33}F_{12} & F_{31}F_{13} + F_{33}F_{11}
\end{bmatrix}
$$

$F_{ij} = \dfrac{\partial x_i}{\partial X_j}$ and $F_{ij} = \dfrac{\partial u_i}{\partial X_j} + \delta_{ij}$

=

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X} + 1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z}$			
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y} + 1\right)$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial v}{\partial Z}$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

=

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z}$			
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial v}{\partial Z}$			
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial w}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

=

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z}$	$\frac{\partial \mathbf{N}}{\partial x}2F_{11}F_{12}+\frac{\partial \mathbf{N}}{\partial y}(F_{11}F_{22}+F_{12}F_{21})+\frac{\partial \mathbf{N}}{\partial z}(F_{31}F_{12}+F_{32}F_{11})$		
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial v}{\partial Z}$			
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial w}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

$$
= \begin{array}{|c|c|c|c|c|c|}
\hline
\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right) & \frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y} & \frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z} & \frac{\partial \mathbf{N}}{\partial x}(F_{11}F_{12}+F_{11}F_{12})+\frac{\partial \mathbf{N}}{\partial y}(F_{11}F_{22}+F_{12}F_{21})+\frac{\partial \mathbf{N}}{\partial z}(F_{31}F_{12}+F_{32}F_{11}) & & \\
\hline
\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X} & \frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right) & \frac{\partial \mathbf{N}}{\partial Z}\frac{\partial v}{\partial Z} & & & \\
\hline
\frac{\partial \mathbf{N}}{\partial X}\frac{\partial w}{\partial X} & \frac{\partial \mathbf{N}}{\partial Y}\frac{\partial w}{\partial Y} & \frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right) & & & \\
\hline
\end{array}
$$

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

$=$

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z}$	$\left(\frac{\partial \mathbf{N}}{\partial x}F_{12}+\frac{\partial \mathbf{N}}{\partial y}F_{22}+\frac{\partial \mathbf{N}}{\partial z}F_{32}\right)F_{11}+\left(\frac{\partial \mathbf{N}}{\partial x}F_{11}+\frac{\partial \mathbf{N}}{\partial y}F_{21}+\frac{\partial \mathbf{N}}{\partial z}F_{31}\right)F_{12}$		
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial v}{\partial Z}$			
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial w}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

=

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z}$	$\left(\frac{\partial \mathbf{N}}{\partial x}\frac{\partial x}{\partial Y}+\frac{\partial \mathbf{N}}{\partial y}\frac{\partial y}{\partial Y}+\frac{\partial \mathbf{N}}{\partial z}\frac{\partial z}{\partial Y}\right)\left(\frac{\partial u}{\partial X}+1\right)+\left(\frac{\partial \mathbf{N}}{\partial x}\frac{\partial x}{\partial X}+\frac{\partial \mathbf{N}}{\partial y}\frac{\partial y}{\partial X}+\frac{\partial \mathbf{N}}{\partial z}\frac{\partial z}{\partial X}\right)\frac{\partial u}{\partial Y}$		
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial v}{\partial Z}$			
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial w}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

$=$

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \mathbf{N}}{\partial X}\frac{\partial u}{\partial Y}$		
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial v}{\partial Z}$			
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial w}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

Summarize by F_{12}
Or summarize by F_{13}

=

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \mathbf{N}}{\partial X}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Y}+\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Z}$	
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial v}{\partial Z}$			
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial w}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

=

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \mathbf{N}}{\partial X}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Y}+\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Z}$	$\frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \mathbf{N}}{\partial X}\frac{\partial u}{\partial Z}$
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial v}{\partial Z}$			
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial w}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \mathbf{N}}{\partial x} & & & \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial z} \\
& \frac{\partial \mathbf{N}}{\partial y} & & \frac{\partial \mathbf{N}}{\partial x} & \frac{\partial \mathbf{N}}{\partial z} & \\
& & \frac{\partial \mathbf{N}}{\partial z} & & \frac{\partial \mathbf{N}}{\partial y} & \frac{\partial \mathbf{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

=

$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Z}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \mathbf{N}}{\partial X}\frac{\partial u}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial u}{\partial Y}+\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial u}{\partial Z}$	$\frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \mathbf{N}}{\partial X}\frac{\partial u}{\partial Z}$
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \mathbf{N}}{\partial Z}\frac{\partial v}{\partial Z}$	$\frac{\partial \mathbf{N}}{\partial X}\left(\frac{\partial v}{\partial Y}+1\right)+\frac{\partial \mathbf{N}}{\partial X}\frac{\partial v}{\partial X}$		
$\frac{\partial \mathbf{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \mathbf{N}}{\partial Y}\frac{\partial w}{\partial Y}$	$\frac{\partial \mathbf{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\
 & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\
 & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

=

$\frac{\partial \boldsymbol{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial u}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial Y}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial u}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial u}{\partial Y}+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial u}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial u}{\partial Z}$
$\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \boldsymbol{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial v}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial X}\left(\frac{\partial v}{\partial Y}+1\right)+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial v}{\partial X}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\left(\frac{\partial v}{\partial Y}+1\right)+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial v}{\partial Z}$	
$\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial w}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\
& \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\
& & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

=

$\frac{\partial \boldsymbol{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial u}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial Y}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial u}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial u}{\partial Y}+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial u}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial u}{\partial Z}$
$\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \boldsymbol{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial v}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial X}\left(\frac{\partial v}{\partial Y}+1\right)+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial v}{\partial X}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\left(\frac{\partial v}{\partial Y}+1\right)+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial v}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial v}{\partial Y}+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial v}{\partial X}$
$\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial w}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$			

B matrix in current state

Transform matrix from real stress to pseudo stress

$$
\begin{bmatrix}
\frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\
& \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\
& & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x}
\end{bmatrix}
\begin{bmatrix}
F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\
F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\
F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\
F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22}+F_{12}F_{21} & F_{12}F_{23}+F_{13}F_{22} & F_{11}F_{23}+F_{13}F_{21} \\
F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32}+F_{22}F_{31} & F_{22}F_{33}+F_{23}F_{32} & F_{21}F_{33}+F_{23}F_{31} \\
F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12}+F_{32}F_{11} & F_{32}F_{13}+F_{33}F_{12} & F_{31}F_{13}+F_{33}F_{11}
\end{bmatrix}
$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} \quad \text{and} \quad F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

=

$\frac{\partial \boldsymbol{N}}{\partial X}\left(\frac{\partial u}{\partial X}+1\right)$	$\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial u}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial u}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial Y}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial u}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial u}{\partial Y}+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial u}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\left(\frac{\partial u}{\partial X}+1\right)+\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial u}{\partial Z}$
$\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial v}{\partial X}$	$\frac{\partial \boldsymbol{N}}{\partial Y}\left(\frac{\partial v}{\partial Y}+1\right)$	$\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial v}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial X}\left(\frac{\partial v}{\partial Y}+1\right)+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial v}{\partial X}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\left(\frac{\partial v}{\partial Y}+1\right)+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial v}{\partial Z}$	$\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial v}{\partial Y}+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial v}{\partial X}$
$\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial w}{\partial X}$	$\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial w}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial Z}\left(\frac{\partial w}{\partial Z}+1\right)$	$\frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial w}{\partial Y}+\frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial w}{\partial X}$	$\frac{\partial \boldsymbol{N}}{\partial Y}\left(\frac{\partial w}{\partial Z}+1\right)+\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial w}{\partial Y}$	$\frac{\partial \boldsymbol{N}}{\partial X}\left(\frac{\partial w}{\partial Z}+1\right)+\frac{\partial \boldsymbol{N}}{\partial Z}\frac{\partial w}{\partial X}$

$$\int_B \left[\frac{\partial w_i}{\partial x_j}\right] : \left[\sigma_{ji}\right] \mathrm{d}v = \left[\boldsymbol{W}_1^{\mathrm{T}} \quad \boldsymbol{W}_2^{\mathrm{T}} \quad \boldsymbol{W}_3^{\mathrm{T}}\right] \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \mathrm{d}v$$

$$\sigma_{11} = F_{1i} S_{ij} F_{1j} = F_{11}S_{11}F_{11} + F_{11}S_{12}F_{12} + F_{11}S_{13}F_{13} + F_{12}S_{21}F_{11} + F_{12}S_{22}F_{12} + F_{12}S_{23}F_{13} + F_{13}S_{31}F_{11} + F_{13}S_{32}F_{12} + F_{13}S_{33}F_{13}$$

$$\sigma_{22} = F_{2i} S_{ij} F_{2j} = F_{21}S_{11}F_{21} + F_{21}S_{12}F_{22} + F_{21}S_{13}F_{23} + F_{22}S_{21}F_{21} + F_{22}S_{22}F_{22} + F_{22}S_{23}F_{23} + F_{23}S_{31}F_{21} + F_{23}S_{32}F_{22} + F_{23}S_{33}F_{23}$$

$$\sigma_{33} = F_{3i} S_{ij} F_{3j} = F_{31}S_{11}F_{31} + F_{31}S_{12}F_{32} + F_{31}S_{13}F_{33} + F_{32}S_{21}F_{31} + F_{32}S_{22}F_{32} + F_{32}S_{23}F_{33} + F_{33}S_{31}F_{31} + F_{33}S_{32}F_{32} + F_{33}S_{33}F_{33}$$

$$\sigma_{12} = F_{1i} S_{ij} F_{2j} = F_{11}S_{11}F_{21} + F_{11}S_{12}F_{22} + F_{11}S_{13}F_{23} + F_{12}S_{21}F_{21} + F_{12}S_{22}F_{22} + F_{12}S_{23}F_{23} + F_{13}S_{31}F_{21} + F_{13}S_{32}F_{22} + F_{13}S_{33}F_{23}$$

$$\sigma_{23} = F_{2i} S_{ij} F_{3j} = F_{21}S_{11}F_{31} + F_{21}S_{12}F_{32} + F_{21}S_{13}F_{33} + F_{22}S_{21}F_{31} + F_{22}S_{22}F_{32} + F_{22}S_{23}F_{33} + F_{23}S_{31}F_{31} + F_{23}S_{32}F_{32} + F_{23}S_{33}F_{33}$$

$$\sigma_{31} = F_{3i} S_{ij} F_{1j} = F_{31}S_{11}F_{11} + F_{31}S_{12}F_{12} + F_{31}S_{13}F_{13} + F_{32}S_{21}F_{11} + F_{32}S_{22}F_{12} + F_{32}S_{23}F_{13} + F_{33}S_{31}F_{11} + F_{33}S_{32}F_{12} + F_{33}S_{33}F_{13}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\ F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\ F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\ F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22} + F_{12}F_{21} & F_{12}F_{23} + F_{13}F_{22} & F_{11}F_{23} + F_{13}F_{21} \\ F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32} + F_{22}F_{31} & F_{22}F_{33} + F_{23}F_{32} & F_{21}F_{33} + F_{23}F_{31} \\ F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12} + F_{32}F_{11} & F_{32}F_{13} + F_{33}F_{12} & F_{31}F_{13} + F_{33}F_{11} \end{bmatrix} \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{Bmatrix}$$

$$\int_b \left[\frac{\partial \boldsymbol{w}_i}{\partial x_j}\right] : [\sigma_{ji}]\,\mathrm{d}v = [\boldsymbol{w}_1^{\mathrm{T}} \quad \boldsymbol{w}_2^{\mathrm{T}} \quad \boldsymbol{w}_3^{\mathrm{T}}] \int_b \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \mathrm{d}v$$

$$F_{ij} = \frac{\partial u_i}{\partial X_j} + \delta_{ij}$$

$$\int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{bmatrix} F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\ F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\ F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\ F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22} + F_{12}F_{21} & F_{12}F_{23} + F_{13}F_{22} & F_{11}F_{23} + F_{13}F_{21} \\ F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32} + F_{22}F_{31} & F_{22}F_{33} + F_{23}F_{32} & F_{21}F_{3\square} + F_{23}F_{31} \\ F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12} + F_{32}F_{11} & F_{32}F_{13} + F_{33}F_{12} & F_{31}F_{13} + F_{33}F_{11} \end{bmatrix} \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{Bmatrix} \mathrm{d}V$$

$$\begin{aligned} &\frac{\partial \boldsymbol{N}}{\partial x}(2F_{11}F_{12}) + \frac{\partial \boldsymbol{N}}{\partial y}(F_{11}F_{22} + F_{12}F_{21}) + \frac{\partial \boldsymbol{N}}{\partial z}(F_{31}F_{12} + F_{32}F_{11}) \\ &\quad = \left(\frac{\partial \boldsymbol{N}}{\partial x}F_{12} + \frac{\partial \boldsymbol{N}}{\partial y}F_{22} + \frac{\partial \boldsymbol{N}}{\partial z}F_{32}\right)F_{11} + \left(\frac{\partial \boldsymbol{N}}{\partial x}F_{11} + \frac{\partial \boldsymbol{N}}{\partial y}F_{21} + \frac{\partial \boldsymbol{N}}{\partial z}F_{31}\right)F_{12} \\ &\quad = \left(\frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial \boldsymbol{N}}{\partial y}\frac{\partial y}{\partial Y} + \frac{\partial \boldsymbol{N}}{\partial z}\frac{\partial z}{\partial Y}\right)\left(\frac{\partial u}{\partial X} + 1\right) + \left(\frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial \boldsymbol{N}}{\partial y}\frac{\partial y}{\partial X} + \frac{\partial \boldsymbol{N}}{\partial z}\frac{\partial z}{\partial X}\right)\frac{\partial u}{\partial Y} = \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial \boldsymbol{N}}{\partial Y}\frac{\partial u}{\partial X} + \frac{\partial \boldsymbol{N}}{\partial X}\frac{\partial u}{\partial Y} \end{aligned}$$

Finally

it becomes

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\ \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\ \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \end{bmatrix} \begin{Bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ s_{12} \\ s_{23} \\ s_{31} \end{Bmatrix} \mathrm{d}V$$

Or

$$\boxed{\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B [\mathbf{B}_\mathrm{L} + \mathbf{B}_\mathrm{N}]\{S_{ij}\}\mathrm{d}V}$$

How to solve?

Loads (vector) → **Stress** (tensor) → **Strain** (tensor) → **Deformation** (vector)

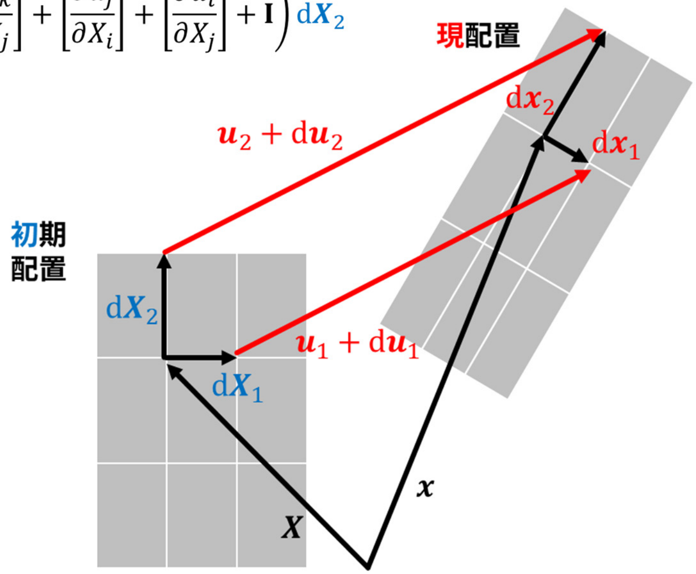
Now we are here

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B [\mathbf{B}]\{S_{ij}\}\mathrm{d}V = \int_B [\mathbf{B}][D]\{E_{ij}\}\mathrm{d}V = \int_B [\mathbf{B}][\mathbf{D}][\overline{\mathbf{B}}]^{\mathrm{T}}\mathrm{d}V \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{Bmatrix}$$

Definition of **Green-Lagrange strain**

$$[E_{ij}] = \frac{1}{2}\left(\left[\frac{\partial u_i}{\partial X_j}\right] + \left[\frac{\partial u_j}{\partial X_i}\right] + \left[\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right]\right)$$

$$\begin{aligned} \mathrm{d}\boldsymbol{x}_1 \cdot \mathrm{d}\boldsymbol{x}_2 &= (\mathbf{F}\mathrm{d}\boldsymbol{X}_1)\cdot(\mathbf{F}\mathrm{d}\boldsymbol{X}_2) \\ &= \mathrm{d}\boldsymbol{X}_1^{\mathrm{T}}\mathbf{U}^2\mathrm{d}\boldsymbol{X}_2 \\ &= \mathrm{d}\boldsymbol{X}_1^{\mathrm{T}}\left(\left[\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right] + \left[\frac{\partial u_j}{\partial X_i}\right] + \left[\frac{\partial u_i}{\partial X_j}\right] + \mathbf{I}\right)\mathrm{d}\boldsymbol{X}_2 \end{aligned}$$



$$E_{11} = \frac{\partial u}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial u}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial v}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial w}{\partial X}$$

$$E_{22} = \frac{\partial v}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial v}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial w}{\partial Y}$$

$$E_{33} = \frac{\partial w}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial u}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial w}{\partial Z}$$

$$E_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\frac{\partial v}{\partial Y} + \frac{\partial w}{\partial X}\frac{\partial w}{\partial Y}\right)$$

$$E_{23} = \frac{1}{2}\left(\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial u}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y}\frac{\partial w}{\partial Z}\right)$$

$$E_{31} = \frac{1}{2}\left(\frac{\partial w}{\partial X} + \frac{\partial u}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Z}\frac{\partial v}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial w}{\partial X}\right)$$

Substitute approximate solutions

$$
\boxed{\begin{aligned} u &= \boldsymbol{N}\cdot\boldsymbol{U} \\ v &= \boldsymbol{N}\cdot\boldsymbol{V} \\ w &= \boldsymbol{N}\cdot\boldsymbol{W} \end{aligned}}
$$

Numerical solutions of deformation

Deformation vector on each discrete node

・・・notice the difference from rotation matrix $\mathbf{U}$ and $\mathbf{V}$

$$
\{E_{ij}\} = \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{Bmatrix} = \begin{Bmatrix}
\frac{\partial u}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial u}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial v}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial w}{\partial X} \\
\frac{\partial v}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial v}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial w}{\partial Y} \\
\frac{\partial w}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial u}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial w}{\partial Z} \\
\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X} + \boxed{\frac{\partial u}{\partial X}\frac{\partial u}{\partial Y}} + \frac{\partial v}{\partial X}\frac{\partial v}{\partial Y} + \frac{\partial w}{\partial X}\frac{\partial w}{\partial Y} \\
\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial u}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial v}{\partial Z} + \frac{\partial w}{\partial Y}\frac{\partial w}{\partial Z} \\
\frac{\partial w}{\partial X} + \frac{\partial u}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial u}{\partial X} + \frac{\partial v}{\partial Z}\frac{\partial v}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial w}{\partial X}
\end{Bmatrix} = \begin{bmatrix}
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} & \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} \\
\frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} \\
\frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} \\
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \left(\frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \left(\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) & \left(\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) \\
\left(\frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \left(\frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \left(\frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) \\
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \left(\frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right) & \left(\frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \left(\frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right)
\end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{Bmatrix}
$$

Shear strain
in engineering field

$$
\frac{\partial u}{\partial X}\frac{\partial u}{\partial Y} = \frac{1}{2}\left(\frac{\partial u}{\partial X}\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial u}{\partial X}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X}\right)\boldsymbol{U}
$$

Non-linear terms

There remains some unknown variables
But let's stop at here now

$\mathbf{B}$

$$\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\ \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \\ \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Z} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial X} & \frac{\partial \boldsymbol{N}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial Y} & \frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}}{\partial Z} \end{bmatrix} \begin{Bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ s_{12} \\ s_{23} \\ s_{31} \end{Bmatrix} \mathrm{d}V$$

A little different in shape・・・

We hope they are in the same shape,
Then the stiffness matrix should be symmetric

$\overline{\mathbf{B}}$

$$\begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} & \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} \\ \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} \\ \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} \\ \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{1}{2}\left(\frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{1}{2}\left(\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{1}{2}\left(\frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{1}{2}\left(\frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) \\ \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{1}{2}\left(\frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{1}{2}\left(\frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right) \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{Bmatrix}$$

$$[E_{ij}] = \frac{1}{2}\left(\left[\frac{\partial u_i}{\partial X_j}\right] + \left[\frac{\partial u_j}{\partial X_i}\right] + \left[\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right]\right)$$ The incremental form is

$$[\Delta E_{ij}] = \frac{1}{2}\left(\left[\frac{\partial \Delta u_i}{\partial X_j}\right] + \left[\frac{\partial \Delta u_j}{\partial X_i}\right] + \left[\frac{\partial \Delta u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right] + \left[\frac{\partial u_k}{\partial X_i}\frac{\partial \Delta u_k}{\partial X_j}\right]\right)$$

$$\begin{Bmatrix} \Delta E_{11} \\ \Delta E_{22} \\ \Delta E_{33} \\ 2\Delta E_{12} \\ 2\Delta E_{23} \\ 2\Delta E_{31} \end{Bmatrix} = \underbrace{\begin{bmatrix}
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} & \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} \\
\frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} \\
\frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} \\
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \left(\frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \left(\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) & \left(\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) \\
\left(\frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \left(\frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} + \left(\frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) \\
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} + \left(\frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right) & \left(\frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \left(\frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} + \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right)
\end{bmatrix}}_{\mathbf{B}} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{Bmatrix}$$

Assumption of **constitutive model**

$$\begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & & & \\ \lambda & \lambda + 2\mu & \lambda & & & \\ \lambda & \lambda & \lambda + 2\mu & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{Bmatrix}$$

2nd Piola-Kirchhoff stress

Green-Lagrange strain

Kind of different from the reality, but it is simple enough

The more accurate the better

Assumption of **constitutive model**

$$\{S_{ij}\} = [\mathbf{D}]\{E_{ij}\}$$

Kind of different from the reality, but it is simple enough

The more accurate the better

$$\begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} = \left[\int_B [\mathbf{B}][\mathbf{D}][\overline{\mathbf{B}}]^{\mathrm{T}} \mathrm{d}V \right] \begin{Bmatrix} U \\ V \\ W \end{Bmatrix} \Rightarrow \boxed{\begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} = \left[\quad \mathbf{K} \quad \right] \begin{Bmatrix} U \\ V \\ W \end{Bmatrix}}$$

There is no **inverse matrix** for **stiffness matrix**

Separate unknown and known (given)

One of load and deformation is known, and the remaining one is unknown

$$\begin{Bmatrix} F_X \\ F_Y \\ F_Z \end{Bmatrix} = \begin{bmatrix} & & \\ & \mathbf{K} & \\ & & \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \end{Bmatrix}$$

$$\begin{Bmatrix} \bar{\bar{F}}_X \\ \bar{F}_X \\ \bar{\bar{F}}_Y \\ \bar{F}_Y \\ \bar{\bar{F}}_Z \\ \bar{F}_Z \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{\bar{\bar{F}}_X\bar{\bar{U}}} & \mathbf{K}_{\bar{\bar{F}}_X\bar{U}} & \mathbf{K}_{\bar{\bar{F}}_X\bar{\bar{V}}} & \mathbf{K}_{\bar{\bar{F}}_X\bar{V}} & \mathbf{K}_{\bar{\bar{F}}_X\bar{\bar{W}}} & \mathbf{K}_{\bar{\bar{F}}_X\bar{W}} \\ \mathbf{K}_{\bar{F}_X\bar{\bar{U}}} & \mathbf{K}_{\bar{F}_X\bar{U}} & \mathbf{K}_{\bar{F}_X\bar{\bar{V}}} & \mathbf{K}_{\bar{F}_X\bar{V}} & \mathbf{K}_{\bar{F}_X\bar{\bar{W}}} & \mathbf{K}_{\bar{F}_X\bar{W}} \\ \mathbf{K}_{\bar{\bar{F}}_Y\bar{\bar{U}}} & \mathbf{K}_{\bar{\bar{F}}_Y\bar{U}} & \mathbf{K}_{\bar{\bar{F}}_Y\bar{\bar{V}}} & \mathbf{K}_{\bar{\bar{F}}_Y\bar{V}} & \mathbf{K}_{\bar{\bar{F}}_Y\bar{\bar{W}}} & \mathbf{K}_{\bar{\bar{F}}_Y\bar{W}} \\ \mathbf{K}_{\bar{F}_Y\bar{\bar{U}}} & \mathbf{K}_{\bar{F}_Y\bar{U}} & \mathbf{K}_{\bar{F}_Y\bar{\bar{V}}} & \mathbf{K}_{\bar{F}_Y\bar{V}} & \mathbf{K}_{\bar{F}_Y\bar{\bar{W}}} & \mathbf{K}_{\bar{F}_Y\bar{W}} \\ \mathbf{K}_{\bar{\bar{F}}_Z\bar{\bar{U}}} & \mathbf{K}_{\bar{\bar{F}}_Z\bar{U}} & \mathbf{K}_{\bar{\bar{F}}_Z\bar{\bar{V}}} & \mathbf{K}_{\bar{\bar{F}}_Z\bar{V}} & \mathbf{K}_{\bar{\bar{F}}_Z\bar{\bar{W}}} & \mathbf{K}_{\bar{\bar{F}}_Z\bar{W}} \\ \mathbf{K}_{\bar{F}_Z\bar{\bar{U}}} & \mathbf{K}_{\bar{F}_Z\bar{U}} & \mathbf{K}_{\bar{F}_Z\bar{\bar{V}}} & \mathbf{K}_{\bar{F}_Z\bar{V}} & \mathbf{K}_{\bar{F}_Z\bar{\bar{W}}} & \mathbf{K}_{\bar{F}_Z\bar{W}} \end{bmatrix} \begin{Bmatrix} \bar{\bar{U}} \\ \bar{U} \\ \bar{\bar{V}} \\ \bar{V} \\ \bar{\bar{W}} \\ \bar{W} \end{Bmatrix}$$

Erase unknown loads

$$
\begin{Bmatrix} \overline{\overline{F}}_X \\ \overline{F}_X \\ \overline{\overline{F}}_Y \\ \overline{F}_Y \\ \overline{\overline{F}}_Z \\ \overline{F}_Z \end{Bmatrix}
=
\begin{bmatrix}
K_{\overline{\overline{F}}_X\overline{\overline{U}}} & K_{\overline{\overline{F}}_X\overline{U}} & K_{\overline{\overline{F}}_X\overline{\overline{V}}} & K_{\overline{\overline{F}}_X\overline{V}} & K_{\overline{\overline{F}}_X\overline{\overline{W}}} & K_{\overline{\overline{F}}_X\overline{W}} \\
K_{\overline{F}_X\overline{\overline{U}}} & K_{\overline{F}_X\overline{U}} & K_{\overline{F}_X\overline{\overline{V}}} & K_{\overline{F}_X\overline{V}} & K_{\overline{F}_X\overline{\overline{W}}} & K_{\overline{F}_X\overline{W}} \\
K_{\overline{\overline{F}}_Y\overline{\overline{U}}} & K_{\overline{\overline{F}}_Y\overline{U}} & K_{\overline{\overline{F}}_Y\overline{\overline{V}}} & K_{\overline{\overline{F}}_Y\overline{V}} & K_{\overline{\overline{F}}_Y\overline{\overline{W}}} & K_{\overline{\overline{F}}_Y\overline{W}} \\
K_{\overline{F}_Y\overline{\overline{U}}} & K_{\overline{F}_Y\overline{U}} & K_{\overline{F}_Y\overline{\overline{V}}} & K_{\overline{F}_Y\overline{V}} & K_{\overline{F}_Y\overline{\overline{W}}} & K_{\overline{F}_Y\overline{W}} \\
K_{\overline{\overline{F}}_Z\overline{\overline{U}}} & K_{\overline{\overline{F}}_Z\overline{U}} & K_{\overline{\overline{F}}_Z\overline{\overline{V}}} & K_{\overline{\overline{F}}_Z\overline{V}} & K_{\overline{\overline{F}}_Z\overline{\overline{W}}} & K_{\overline{\overline{F}}_Z\overline{W}} \\
K_{\overline{F}_Z\overline{\overline{U}}} & K_{\overline{F}_Z\overline{U}} & K_{\overline{F}_Z\overline{\overline{V}}} & K_{\overline{F}_Z\overline{V}} & K_{\overline{F}_Z\overline{\overline{W}}} & K_{\overline{F}_Z\overline{W}}
\end{bmatrix}
\begin{Bmatrix} \overline{\overline{U}} \\ \overline{U} \\ \overline{\overline{V}} \\ \overline{V} \\ \overline{\overline{W}} \\ \overline{W} \end{Bmatrix}
$$

$$
\begin{Bmatrix} \overline{F}_X \\ \overline{F}_Y \\ \overline{F}_Z \end{Bmatrix}
=
\begin{bmatrix}
K_{\overline{F}_X\overline{\overline{U}}} & K_{\overline{F}_X\overline{U}} & K_{\overline{F}_X\overline{\overline{V}}} & K_{\overline{F}_X\overline{V}} & K_{\overline{F}_X\overline{\overline{W}}} & K_{\overline{F}_X\overline{W}} \\
K_{\overline{F}_Y\overline{\overline{U}}} & K_{\overline{F}_Y\overline{U}} & K_{\overline{F}_Y\overline{\overline{V}}} & K_{\overline{F}_Y\overline{V}} & K_{\overline{F}_Y\overline{\overline{W}}} & K_{\overline{F}_Y\overline{W}} \\
K_{\overline{F}_Z\overline{\overline{U}}} & K_{\overline{F}_Z\overline{U}} & K_{\overline{F}_Z\overline{\overline{V}}} & K_{\overline{F}_Z\overline{V}} & K_{\overline{F}_Z\overline{\overline{W}}} & K_{\overline{F}_Z\overline{W}}
\end{bmatrix}
\begin{Bmatrix} \overline{\overline{U}} \\ \overline{U} \\ \overline{\overline{V}} \\ \overline{V} \\ \overline{\overline{W}} \\ \overline{W} \end{Bmatrix}
$$

Move known deformation

$$\begin{Bmatrix} \bar{F}_X \\ \bar{F}_Y \\ \bar{F}_Z \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{\bar{F}_X \bar{\bar{U}}} & \mathbf{K}_{\bar{F}_X \bar{U}} & \mathbf{K}_{\bar{F}_X \bar{\bar{V}}} & \mathbf{K}_{\bar{F}_X \bar{V}} & \mathbf{K}_{\bar{F}_X \bar{\bar{W}}} & \mathbf{K}_{\bar{F}_X \bar{W}} \\ \mathbf{K}_{\bar{F}_Y \bar{\bar{U}}} & \mathbf{K}_{\bar{F}_Y \bar{U}} & \mathbf{K}_{\bar{F}_Y \bar{\bar{V}}} & \mathbf{K}_{\bar{F}_Y \bar{V}} & \mathbf{K}_{\bar{F}_Y \bar{\bar{W}}} & \mathbf{K}_{\bar{F}_Y \bar{W}} \\ \mathbf{K}_{\bar{F}_Z \bar{\bar{U}}} & \mathbf{K}_{\bar{F}_Z \bar{U}} & \mathbf{K}_{\bar{F}_Z \bar{\bar{V}}} & \mathbf{K}_{\bar{F}_Z \bar{V}} & \mathbf{K}_{\bar{F}_Z \bar{\bar{W}}} & \mathbf{K}_{\bar{F}_Z \bar{W}} \end{bmatrix} \begin{Bmatrix} \bar{\bar{U}} \\ \bar{U} \\ \bar{\bar{V}} \\ \bar{V} \\ \bar{\bar{W}} \\ \bar{W} \end{Bmatrix}$$

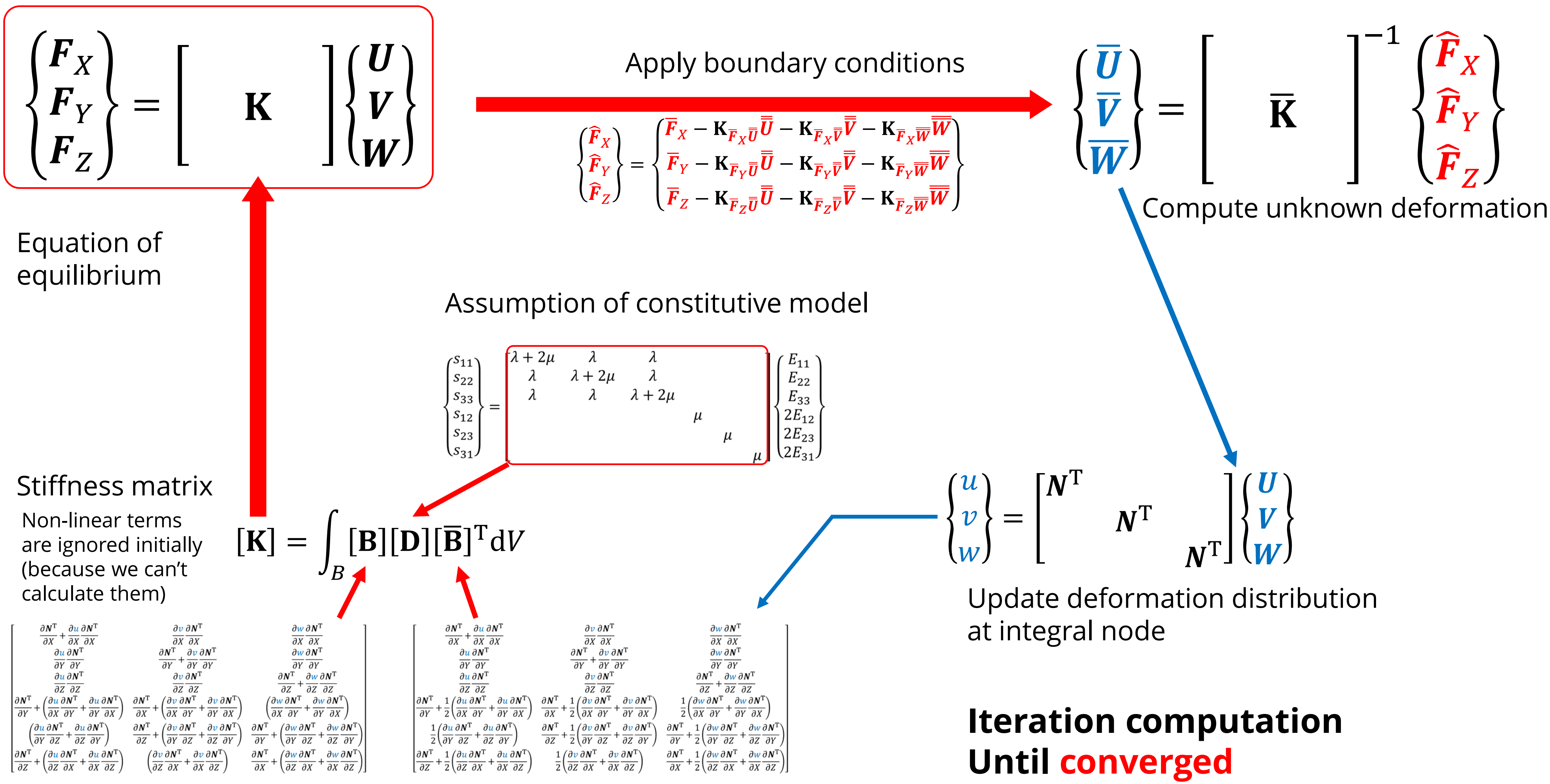
$$\begin{Bmatrix} \bar{F}_X - \mathbf{K}_{\bar{F}_X \bar{\bar{U}}} \bar{\bar{U}} - \mathbf{K}_{\bar{F}_X \bar{\bar{V}}} \bar{\bar{V}} - \mathbf{K}_{\bar{F}_X \bar{\bar{W}}} \bar{\bar{W}} \\ \bar{F}_Y - \mathbf{K}_{\bar{F}_Y \bar{\bar{U}}} \bar{\bar{U}} - \mathbf{K}_{\bar{F}_Y \bar{\bar{V}}} \bar{\bar{V}} - \mathbf{K}_{\bar{F}_Y \bar{\bar{W}}} \bar{\bar{W}} \\ \bar{F}_Z - \mathbf{K}_{\bar{F}_Z \bar{\bar{U}}} \bar{\bar{U}} - \mathbf{K}_{\bar{F}_Z \bar{\bar{V}}} \bar{\bar{V}} - \mathbf{K}_{\bar{F}_Z \bar{\bar{W}}} \bar{\bar{W}} \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{\bar{F}_X \bar{U}} & \mathbf{K}_{\bar{F}_X \bar{V}} & \mathbf{K}_{\bar{F}_X \bar{W}} \\ \mathbf{K}_{\bar{F}_Y \bar{U}} & \mathbf{K}_{\bar{F}_Y \bar{V}} & \mathbf{K}_{\bar{F}_Y \bar{W}} \\ \mathbf{K}_{\bar{F}_Z \bar{U}} & \mathbf{K}_{\bar{F}_Z \bar{V}} & \mathbf{K}_{\bar{F}_Z \bar{W}} \end{bmatrix} \begin{Bmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{Bmatrix}$$

Update the definition

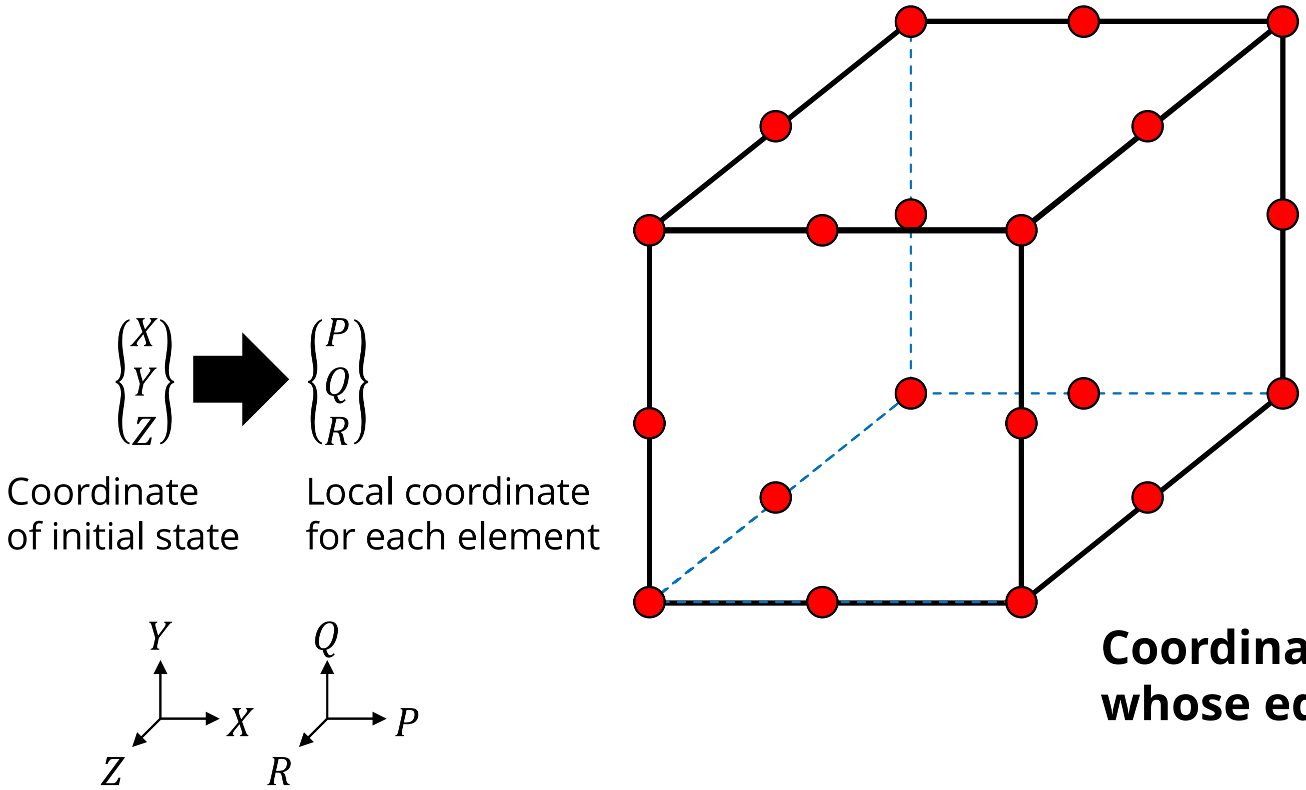
$$\begin{Bmatrix} \bar{F}_X - \mathbf{K}_{\bar{F}_X \bar{\bar{U}}} \bar{\bar{U}} - \mathbf{K}_{\bar{F}_X \bar{\bar{V}}} \bar{\bar{V}} - \mathbf{K}_{\bar{F}_X \overline{\overline{W}}} \overline{\overline{W}} \\ \bar{F}_Y - \mathbf{K}_{\bar{F}_Y \bar{\bar{U}}} \bar{\bar{U}} - \mathbf{K}_{\bar{F}_Y \bar{\bar{V}}} \bar{\bar{V}} - \mathbf{K}_{\bar{F}_Y \overline{\overline{W}}} \overline{\overline{W}} \\ \bar{F}_Z - \mathbf{K}_{\bar{F}_Z \bar{\bar{U}}} \bar{\bar{U}} - \mathbf{K}_{\bar{F}_Z \bar{\bar{V}}} \bar{\bar{V}} - \mathbf{K}_{\bar{F}_Z \overline{\overline{W}}} \overline{\overline{W}} \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{\bar{F}_X \bar{U}} & \mathbf{K}_{\bar{F}_X \bar{V}} & \mathbf{K}_{\bar{F}_X \overline{W}} \\ \mathbf{K}_{\bar{F}_Y \bar{U}} & \mathbf{K}_{\bar{F}_Y \bar{V}} & \mathbf{K}_{\bar{F}_Y \overline{W}} \\ \mathbf{K}_{\bar{F}_Z \bar{U}} & \mathbf{K}_{\bar{F}_Z \bar{V}} & \mathbf{K}_{\bar{F}_Z \overline{W}} \end{bmatrix} \begin{Bmatrix} \bar{U} \\ \bar{V} \\ \overline{W} \end{Bmatrix}$$

$$\begin{Bmatrix} \hat{F}_X \\ \hat{F}_Y \\ \hat{F}_Z \end{Bmatrix} = \begin{bmatrix} \bar{\mathbf{K}} \end{bmatrix} \begin{Bmatrix} \bar{U} \\ \bar{V} \\ \overline{W} \end{Bmatrix}$$

Applying **boundary condition** let us to **inverse** the matrix**!**

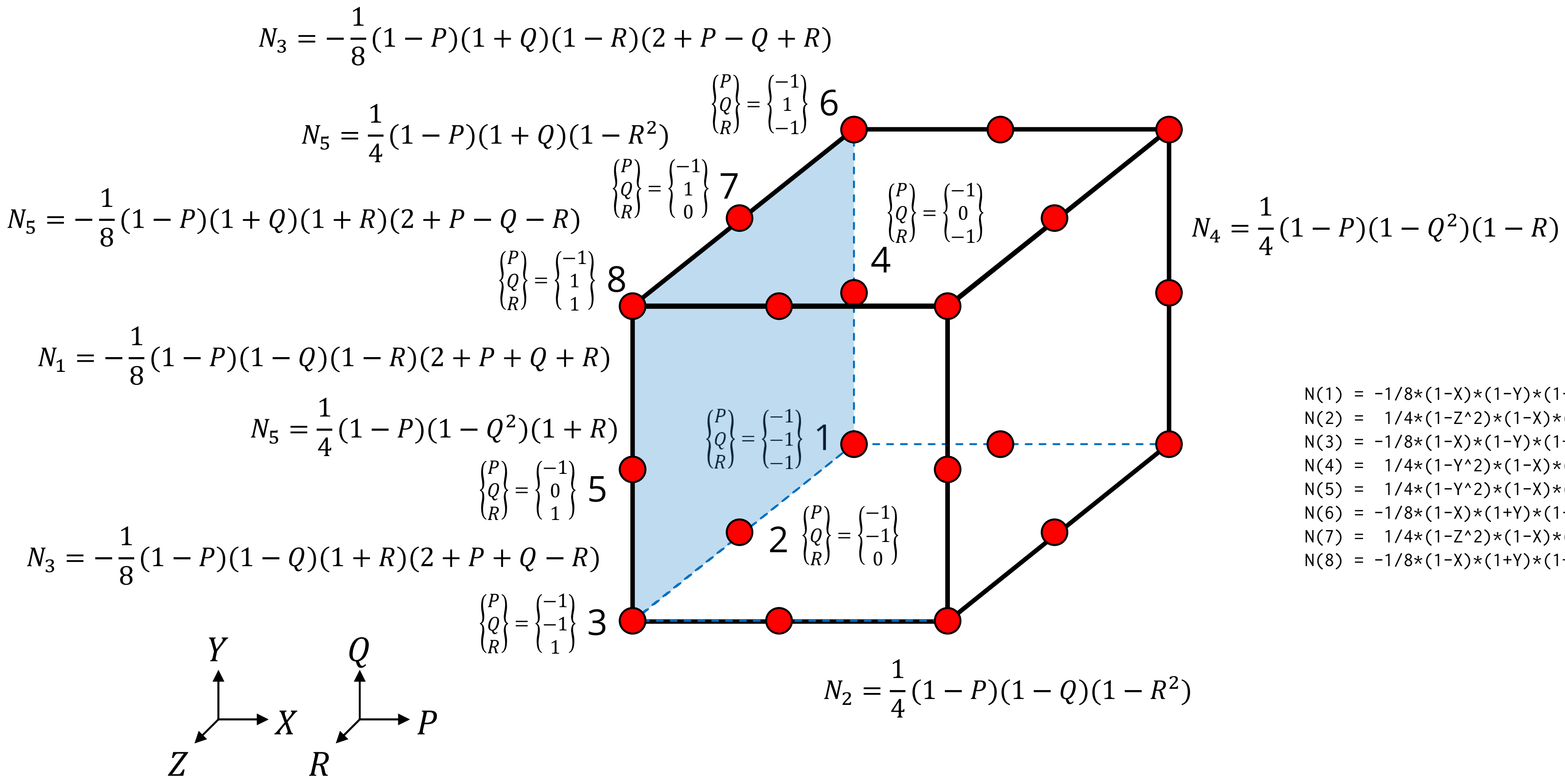

Equation of equilibrium
Apply boundary conditions
Compute unknown deformation
Assumption of constitutive model
Stiffness matrix
Non-linear terms are ignored initially (because we can't calculate them)
Update deformation distribution at integral node
Iteration computation
Until converged

Shape function : $\boldsymbol{N}^{\mathrm{T}}$



Coordinate transformation into a cubic whose edges are defined from -1 to 1

Shape function：$\boldsymbol{N}^{\mathrm{T}}$



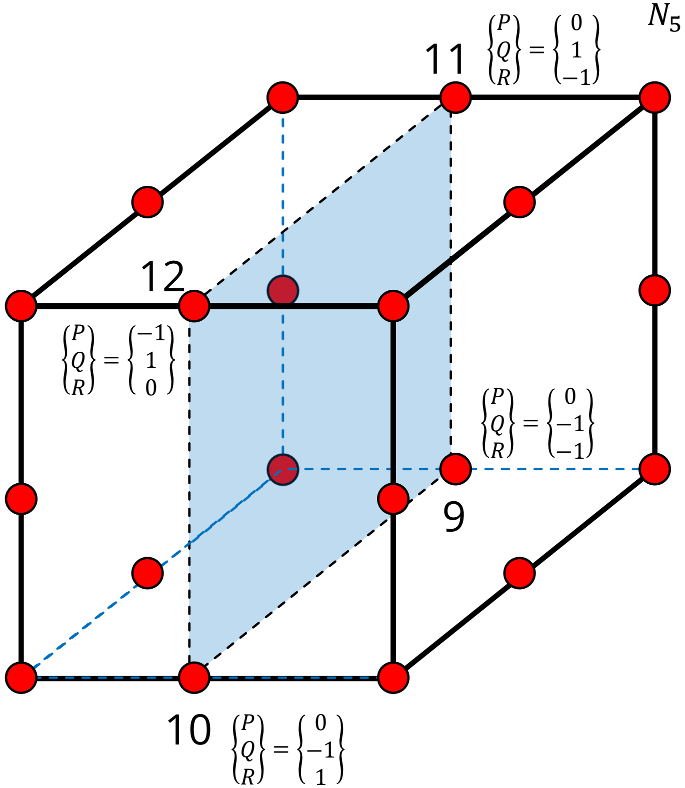

```
N(1) = -1/8*(1-X)*(1-Y)*(1-Z)*(2+X+Y+Z);
N(2) =  1/4*(1-Z^2)*(1-X)*(1-Y);
N(3) = -1/8*(1-X)*(1-Y)*(1+Z)*(2+X+Y-Z);
N(4) =  1/4*(1-Y^2)*(1-X)*(1-Z);
N(5) =  1/4*(1-Y^2)*(1-X)*(1+Z);
N(6) = -1/8*(1-X)*(1+Y)*(1-Z)*(2+X-Y+Z);
N(7) =  1/4*(1-Z^2)*(1-X)*(1+Y);
N(8) = -1/8*(1-X)*(1+Y)*(1+Z)*(2+X-Y-Z);
```

```
N(1) = -1/8*(1-X)*(1-Y)*(1-Z)*(2+X+Y+Z);
N(2) =  1/4*(1-Z^2)*(1-X)*(1-Y);
N(3) = -1/8*(1-X)*(1-Y)*(1+Z)*(2+X+Y-Z);
N(4) =  1/4*(1-Y^2)*(1-X)*(1-Z);
N(5) =  1/4*(1-Y^2)*(1-X)*(1+Z);
N(6) = -1/8*(1-X)*(1+Y)*(1-Z)*(2+X-Y+Z);
N(7) =  1/4*(1-Z^2)*(1-X)*(1+Y);
N(8) = -1/8*(1-X)*(1+Y)*(1+Z)*(2+X-Y-Z);

N(9) =  1/4*(1-X^2)*(1-Y)*(1-Z);
N(10)=  1/4*(1-X^2)*(1-Y)*(1+Z);
N(11)=  1/4*(1-X^2)*(1+Y)*(1-Z);
N(12)=  1/4*(1-X^2)*(1+Y)*(1+Z);
```

Shape function：$\boldsymbol{N}^{\mathrm{T}}$

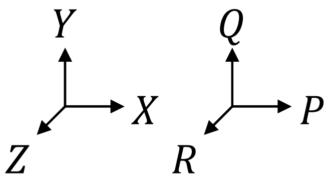
$$N_5 = \frac{1}{4}(1-P^2)(1+Q)(1-R)$$

$$N_{12} = \frac{1}{4}(1-P^2)(1+Q)(1+R)$$



$$N_9 = \frac{1}{4}(1-P^2)(1-Q)(1-R)$$

$$N_4 = \frac{1}{4}(1-P^2)(1-Q)(1+R)$$

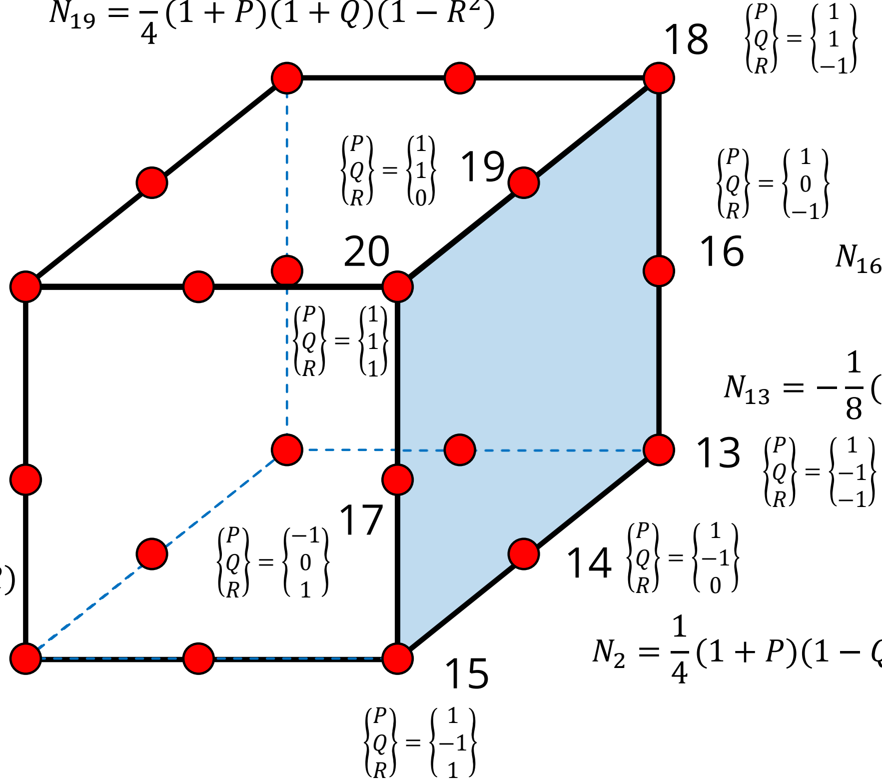


```
N(1) = -1/8*(1-X)*(1-Y)*(1-Z)*(2+X+Y+Z);
N(2) =  1/4*(1-Z^2)*(1-X)*(1-Y);
N(3) = -1/8*(1-X)*(1-Y)*(1+Z)*(2+X+Y-Z);
N(4) =  1/4*(1-Y^2)*(1-X)*(1-Z);
N(5) =  1/4*(1-Y^2)*(1-X)*(1+Z);
N(6) = -1/8*(1-X)*(1+Y)*(1-Z)*(2+X-Y+Z);
N(7) =  1/4*(1-Z^2)*(1-X)*(1+Y);
N(8) = -1/8*(1-X)*(1+Y)*(1+Z)*(2+X-Y-Z);

N(9) =  1/4*(1-X^2)*(1-Y)*(1-Z);
N(10)=  1/4*(1-X^2)*(1-Y)*(1+Z);
N(11)=  1/4*(1-X^2)*(1+Y)*(1-Z);
N(12)=  1/4*(1-X^2)*(1+Y)*(1+Z);

N(13)= -1/8*(1+X)*(1-Y)*(1-Z)*(2-X+Y+Z);
N(14)=  1/4*(1-Z^2)*(1+X)*(1-Y);
N(15)= -1/8*(1+X)*(1-Y)*(1+Z)*(2-X+Y-Z);
N(16)=  1/4*(1-Y^2)*(1+X)*(1-Z);
N(17)=  1/4*(1-Y^2)*(1+X)*(1+Z);
N(18)= -1/8*(1+X)*(1+Y)*(1-Z)*(2-X-Y+Z);
N(19)=  1/4*(1-Z^2)*(1+X)*(1+Y);
N(20)= -1/8*(1+X)*(1+Y)*(1+Z)*(2-X-Y-Z);
```

Shape function：$\boldsymbol{N}^{\mathrm{T}}$

$$N_{18} = -\frac{1}{8}(1+P)(1+Q)(1-R)(2-P-Q+R)$$

$$N_{19} = \frac{1}{4}(1+P)(1+Q)(1-R^2)$$



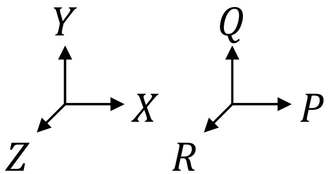
$$N_{16} = \frac{1}{4}(1+P)(1-Q^2)(1-R)$$

$$N_{13} = -\frac{1}{8}(1+P)(1-Q)(1-R)(2-P+Q+R)$$

$$N_{20} = -\frac{1}{8}(1+P)(1+Q)(1+R)(2-P-Q-R)$$

$$N_{17} = \frac{1}{4}(1+P)(1-Q^2)(1+R)$$

$$N_2 = \frac{1}{4}(1+P)(1-Q)(1-R^2)$$



$$N_{15} = -\frac{1}{8}(1+P)(1-Q)(1+R)(2-P+Q-R)$$

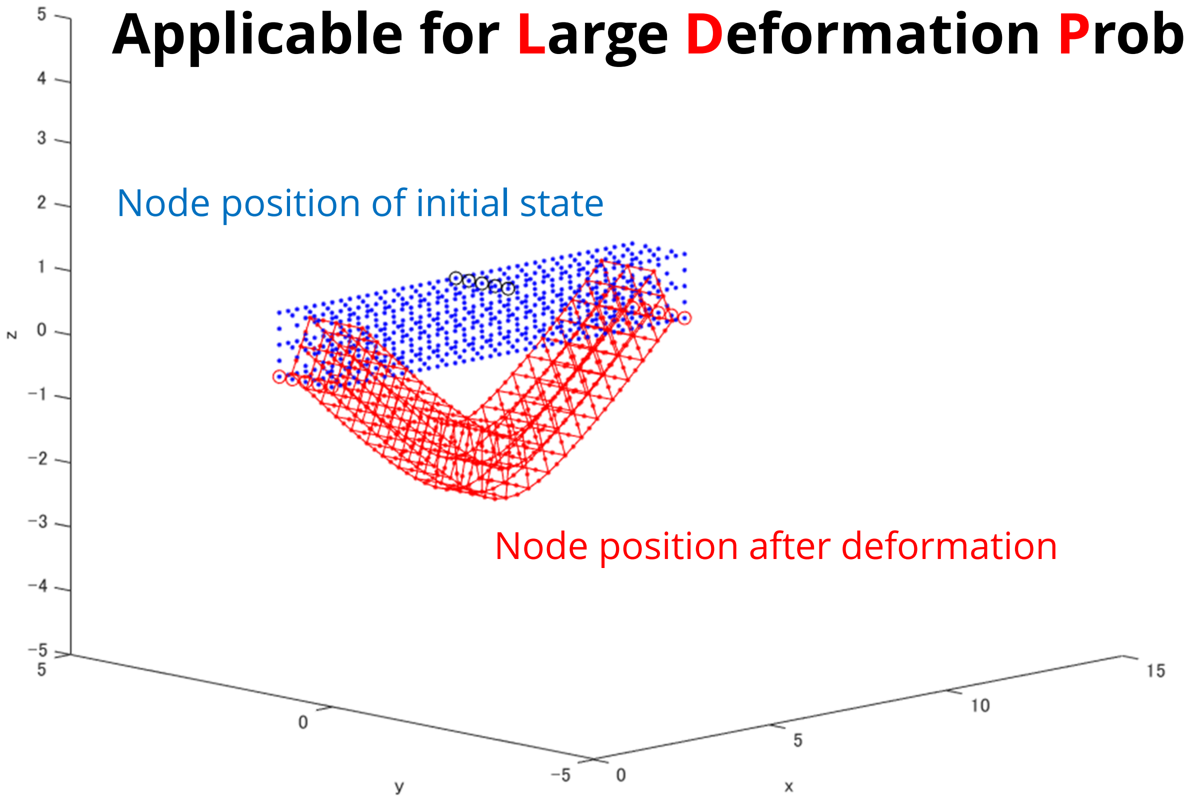
Locking happens if 1^{st} –order element is used

(without deformation)

(strategy)

- **use of higher-order elements**
- use of reduced integrals

Applicable for Large Deformation Problem



$$\begin{bmatrix}
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}+\frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} & \frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} & \frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X} \\
\frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}+\frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} & \frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y} \\
\frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} & \frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}+\frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z} \\
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}+\left(\frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}+\frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}+\left(\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}+\frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) & \left(\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}+\frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}\right) \\
\left(\frac{\partial u}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}+\frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}+\left(\frac{\partial v}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}+\frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}+\left(\frac{\partial w}{\partial Y}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}+\frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Y}\right) \\
\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}+\left(\frac{\partial u}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}+\frac{\partial u}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right) & \left(\frac{\partial v}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}+\frac{\partial v}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right) & \frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}+\left(\frac{\partial w}{\partial Z}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial X}+\frac{\partial w}{\partial X}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial Z}\right)
\end{bmatrix}$$

$$\{\boldsymbol{f}\} = \int_B [\mathbf{B}]\{\boldsymbol{\sigma}\}\,\mathrm{d}v$$

シミュレーションには
有限要素法を用いる。

運動方程式

$$\boldsymbol{r} = \boldsymbol{0}$$

重み付き残差法

$$\int_B \boldsymbol{w} \cdot \boldsymbol{r} \, \mathrm{d}v = 0$$

固体の有限要素式

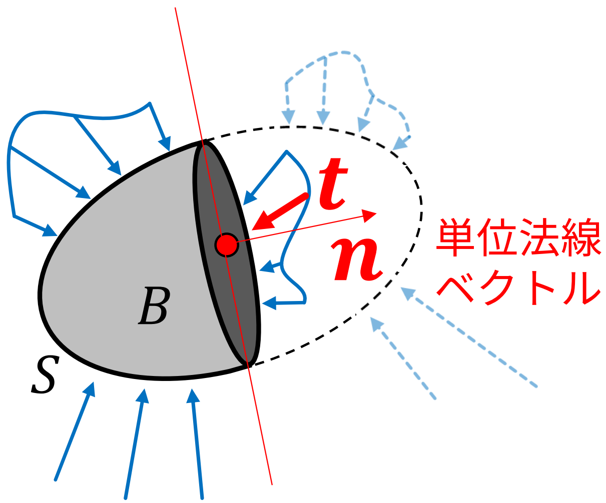
$$\int_s \{\boldsymbol{\omega}\} \cdot \{\boldsymbol{t}\}\, \mathrm{d}a = \int_b \left[\frac{\partial \omega_i}{\partial x_j}\right] : \left[\sigma_{ji}\right] \mathrm{d}v$$

$$[\boldsymbol{\sigma}] = \frac{1}{J} \boldsymbol{F} [\mathbf{S}] \mathbf{F}^{\mathrm{T}}$$

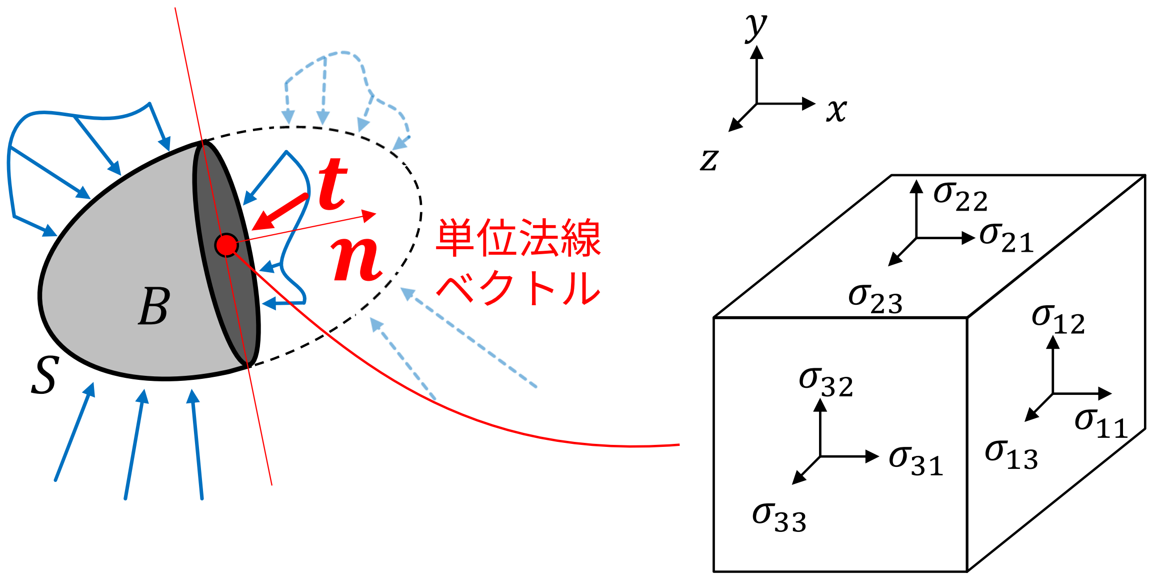
$$\int_S \left[\ S\right]^{\mathrm{T}} \{n_j\}\, \mathrm{d}a = \mathbf{0}$$

$$\int_s \left[\quad\right]^{\mathrm{T}} \{\quad\}\, \mathrm{d}a = \mathbf{0}$$

応力ベクトルは
応力状態と切断面から導く



各点の応力状態は
切断面に影響されない



$$\begin{Bmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_2}{\partial x_1} & \frac{\partial X_3}{\partial x_1} \\ \frac{\partial X_1}{\partial x_2} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_3}{\partial x_2} \\ \frac{\partial X_1}{\partial x_3} & \frac{\partial X_2}{\partial x_3} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} \begin{Bmatrix} \partial/\partial X_1 \\ \partial/\partial X_2 \\ \partial/\partial X_3 \end{Bmatrix}$$

変形勾配テンソル

$$\{\mathrm{d}\boldsymbol{x}\} = \left[\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{X}}\right]\{\mathrm{d}\boldsymbol{X}\}$$

$$\mathbf{F} = \mathbf{R}\mathbf{U}$$

$$\begin{aligned}\mathbf{F}^{\mathrm{T}}\mathbf{F} &= \mathbf{U}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{U} \\ &= \mathbf{U}^{\mathrm{T}}\mathbf{U} \\ &= \mathbf{U}^{2}\end{aligned}$$

$$\mathbf{F} = \mathbf{V}\mathbf{R}$$

$$\begin{aligned}\mathbf{F}\mathbf{F}^{\mathrm{T}} &= \mathbf{V}\mathbf{R}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}} \\ &= \mathbf{V}\mathbf{V}^{\mathrm{T}} \\ &= \mathbf{V}^{2}\end{aligned}$$

$$\int_B \left[\frac{\partial w_i}{\partial x_j}\right] : [\sigma_{ji}] \mathrm{d}v = [\boldsymbol{W}_1^{\mathrm{T}} \quad \boldsymbol{W}_2^{\mathrm{T}} \quad \boldsymbol{W}_3^{\mathrm{T}}] \int_B \begin{bmatrix} \frac{\partial \boldsymbol{N}}{\partial x} & & & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial z} \\ & \frac{\partial \boldsymbol{N}}{\partial y} & & \frac{\partial \boldsymbol{N}}{\partial x} & \frac{\partial \boldsymbol{N}}{\partial z} & \\ & & \frac{\partial \boldsymbol{N}}{\partial z} & & \frac{\partial \boldsymbol{N}}{\partial y} & \frac{\partial \boldsymbol{N}}{\partial x} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} \mathrm{d}v$$

$$\sigma_{11} = F_{1i}S_{ij}F_{1j} = F_{11}S_{11}F_{11} + F_{11}S_{12}F_{12} + F_{11}S_{13}F_{13} + F_{12}S_{21}F_{11} + F_{12}S_{22}F_{12} + F_{12}S_{23}F_{13} + F_{13}S_{31}F_{11} + F_{13}S_{32}F_{12} + F_{13}S_{33}F_{13}$$

$$\sigma_{22} = F_{2i}S_{ij}F_{2j} = F_{21}S_{11}F_{21} + F_{21}S_{12}F_{22} + F_{21}S_{13}F_{23} + F_{22}S_{21}F_{21} + F_{22}S_{22}F_{22} + F_{22}S_{23}F_{23} + F_{23}S_{31}F_{21} + F_{23}S_{32}F_{22} + F_{23}S_{33}F_{23}$$

$$\sigma_{33} = F_{3i}S_{ij}F_{3j} = F_{31}S_{11}F_{31} + F_{31}S_{12}F_{32} + F_{31}S_{13}F_{33} + F_{32}S_{21}F_{31} + F_{32}S_{22}F_{32} + F_{32}S_{23}F_{33} + F_{33}S_{31}F_{31} + F_{33}S_{32}F_{32} + F_{33}S_{33}F_{33}$$

$$\sigma_{12} = F_{1i}S_{ij}F_{2j} = F_{11}S_{11}F_{21} + F_{11}S_{12}F_{22} + F_{11}S_{13}F_{23} + F_{12}S_{21}F_{21} + F_{12}S_{22}F_{22} + F_{12}S_{23}F_{23} + F_{13}S_{31}F_{21} + F_{13}S_{32}F_{22} + F_{13}S_{33}F_{23}$$

$$\sigma_{23} = F_{2i}S_{ij}F_{3j} = F_{21}S_{11}F_{31} + F_{21}S_{12}F_{32} + F_{21}S_{13}F_{33} + F_{22}S_{21}F_{31} + F_{22}S_{22}F_{32} + F_{22}S_{23}F_{33} + F_{23}S_{31}F_{31} + F_{23}S_{32}F_{32} + F_{23}S_{33}F_{33}$$

$$\sigma_{31} = F_{3i}S_{ij}F_{1j} = F_{31}S_{11}F_{11} + F_{31}S_{12}F_{12} + F_{31}S_{13}F_{13} + F_{32}S_{21}F_{11} + F_{32}S_{22}F_{12} + F_{32}S_{23}F_{13} + F_{33}S_{31}F_{11} + F_{33}S_{32}F_{12} + F_{33}S_{33}F_{13}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\ F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\ F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\ F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22} + F_{12}F_{21} & F_{12}F_{23} + F_{13}F_{22} & F_{11}F_{23} + F_{13}F_{21} \\ F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32} + F_{22}F_{31} & F_{22}F_{33} + F_{23}F_{32} & F_{21}F_{33} + F_{23}F_{31} \\ F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12} + F_{32}F_{11} & F_{32}F_{13} + F_{33}F_{12} & F_{31}F_{13} + F_{33}F_{11} \end{bmatrix} \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{Bmatrix}$$

$$\left\{\frac{\partial \omega}{\partial X_i}\right\} = \left[\frac{\partial x_j}{\partial X_i}\right]\left\{\frac{\partial \omega}{\partial x_j}\right\}$$

$$\frac{\partial \omega}{\partial X} = \frac{\partial x}{\partial X}\frac{\partial \omega}{\partial x} + \frac{\partial y}{\partial X}\frac{\partial \omega}{\partial y} + \frac{\partial z}{\partial X}\frac{\partial \omega}{\partial z}$$

$$\frac{\partial \omega}{\partial Y} = \frac{\partial x}{\partial Y}\frac{\partial \omega}{\partial x} + \frac{\partial y}{\partial Y}\frac{\partial \omega}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial \omega}{\partial z}$$

$$\frac{\partial \omega}{\partial Z} = \frac{\partial x}{\partial Z}\frac{\partial \omega}{\partial x} + \frac{\partial y}{\partial Z}\frac{\partial \omega}{\partial y} + \frac{\partial z}{\partial Z}\frac{\partial \omega}{\partial z}$$

$$\begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial y}{\partial X} & \frac{\partial z}{\partial X} \\ \frac{\partial x}{\partial Y} & \frac{\partial y}{\partial Y} & \frac{\partial z}{\partial Y} \\ \frac{\partial x}{\partial Z} & \frac{\partial y}{\partial Z} & \frac{\partial z}{\partial Z} \end{bmatrix}$$

$$\begin{array}{cc} \boldsymbol{F}^{-1} & \boldsymbol{F} \\ \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial y}{\partial X} & \frac{\partial z}{\partial X} \\ \frac{\partial x}{\partial Y} & \frac{\partial y}{\partial Y} & \frac{\partial z}{\partial Y} \\ \frac{\partial x}{\partial Z} & \frac{\partial y}{\partial Z} & \frac{\partial z}{\partial Z} \end{bmatrix} & \begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial Y}{\partial x} & \frac{\partial Z}{\partial x} \\ \frac{\partial X}{\partial y} & \frac{\partial Y}{\partial y} & \frac{\partial Z}{\partial y} \\ \frac{\partial X}{\partial z} & \frac{\partial Y}{\partial z} & \frac{\partial Z}{\partial z} \end{bmatrix} \end{array}$$

$$\frac{\partial x}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial y}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial X}\frac{\partial X}{\partial z} = \frac{\partial X}{\partial X} = 1$$

$$\frac{\partial x}{\partial X}\frac{\partial Y}{\partial x} + \frac{\partial y}{\partial X}\frac{\partial Y}{\partial y} + \frac{\partial z}{\partial X}\frac{\partial Y}{\partial z} = \frac{\partial Y}{\partial X} = 0$$

未知と既知（given）で分類

荷重と変位はいずれかが未知の時、対応するもう一方は既知

$$
\begin{Bmatrix} \boldsymbol{F}_X \\ \boldsymbol{F}_Y \\ \boldsymbol{F}_Z \end{Bmatrix} = \begin{bmatrix} & \mathbf{K} & \end{bmatrix} \begin{Bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{W} \end{Bmatrix}
$$

$$
\begin{Bmatrix} \overline{\overline{\boldsymbol{F}}}_X \\ \overline{\boldsymbol{F}}_X \\ \overline{\overline{\boldsymbol{F}}}_Y \\ \overline{\boldsymbol{F}}_Y \\ \overline{\overline{\boldsymbol{F}}}_Z \\ \overline{\boldsymbol{F}}_Z \end{Bmatrix} = \begin{bmatrix}
\mathbf{K}_{\overline{\overline{F}}_X\overline{\overline{U}}} & \mathbf{K}_{\overline{\overline{F}}_X\overline{U}} & \mathbf{K}_{\overline{\overline{F}}_X\overline{\overline{V}}} & \mathbf{K}_{\overline{\overline{F}}_X\overline{V}} & \mathbf{K}_{\overline{\overline{F}}_X\overline{\overline{W}}} & \mathbf{K}_{\overline{\overline{F}}_X\overline{W}} \\
\mathbf{K}_{\overline{F}_X\overline{\overline{U}}} & \mathbf{K}_{\overline{F}_X\overline{U}} & \mathbf{K}_{\overline{F}_X\overline{\overline{V}}} & \mathbf{K}_{\overline{F}_X\overline{V}} & \mathbf{K}_{\overline{F}_X\overline{\overline{W}}} & \mathbf{K}_{\overline{F}_X\overline{W}} \\
\mathbf{K}_{\overline{\overline{F}}_Y\overline{\overline{U}}} & \mathbf{K}_{\overline{\overline{F}}_Y\overline{U}} & \mathbf{K}_{\overline{\overline{F}}_Y\overline{\overline{V}}} & \mathbf{K}_{\overline{\overline{F}}_Y\overline{V}} & \mathbf{K}_{\overline{\overline{F}}_Y\overline{\overline{W}}} & \mathbf{K}_{\overline{\overline{F}}_Y\overline{W}} \\
\mathbf{K}_{\overline{F}_Y\overline{\overline{U}}} & \mathbf{K}_{\overline{F}_Y\overline{U}} & \mathbf{K}_{\overline{F}_Y\overline{\overline{V}}} & \mathbf{K}_{\overline{F}_Y\overline{V}} & \mathbf{K}_{\overline{F}_Y\overline{\overline{W}}} & \mathbf{K}_{\overline{F}_Y\overline{W}} \\
\mathbf{K}_{\overline{\overline{F}}_Z\overline{\overline{U}}} & \mathbf{K}_{\overline{\overline{F}}_Z\overline{U}} & \mathbf{K}_{\overline{\overline{F}}_Z\overline{\overline{V}}} & \mathbf{K}_{\overline{\overline{F}}_Z\overline{V}} & \mathbf{K}_{\overline{\overline{F}}_Z\overline{\overline{W}}} & \mathbf{K}_{\overline{\overline{F}}_Z\overline{W}} \\
\mathbf{K}_{\overline{F}_Z\overline{\overline{U}}} & \mathbf{K}_{\overline{F}_Z\overline{U}} & \mathbf{K}_{\overline{F}_Z\overline{\overline{V}}} & \mathbf{K}_{\overline{F}_Z\overline{V}} & \mathbf{K}_{\overline{F}_Z\overline{\overline{W}}} & \mathbf{K}_{\overline{F}_Z\overline{W}}
\end{bmatrix} \begin{Bmatrix} \overline{\overline{\boldsymbol{U}}} \\ \overline{\boldsymbol{U}} \\ \overline{\overline{\boldsymbol{V}}} \\ \overline{\boldsymbol{V}} \\ \overline{\overline{\boldsymbol{W}}} \\ \overline{\boldsymbol{W}} \end{Bmatrix}
$$

荷重が未知のものは消す

$$
\begin{Bmatrix}
\overline{\overline{F}}_X \\ \overline{F}_X \\ \overline{\overline{F}}_Y \\ \overline{F}_Y \\ \overline{\overline{F}}_Z \\ \overline{F}_Z
\end{Bmatrix}
=
\begin{bmatrix}
K_{\overline{\overline{F}}_X \overline{\overline{U}}} & K_{\overline{\overline{F}}_X \overline{U}} & K_{\overline{\overline{F}}_X \overline{\overline{V}}} & K_{\overline{\overline{F}}_X \overline{V}} & K_{\overline{\overline{F}}_X \overline{\overline{W}}} & K_{\overline{\overline{F}}_X \overline{W}} \\
K_{\overline{F}_X \overline{\overline{U}}} & K_{\overline{F}_X \overline{U}} & K_{\overline{F}_X \overline{\overline{V}}} & K_{\overline{F}_X \overline{V}} & K_{\overline{F}_X \overline{\overline{W}}} & K_{\overline{F}_X \overline{W}} \\
K_{\overline{\overline{F}}_Y \overline{\overline{U}}} & K_{\overline{\overline{F}}_Y \overline{U}} & K_{\overline{\overline{F}}_Y \overline{\overline{V}}} & K_{\overline{\overline{F}}_Y \overline{V}} & K_{\overline{\overline{F}}_Y \overline{\overline{W}}} & K_{\overline{\overline{F}}_Y \overline{W}} \\
K_{\overline{F}_Y \overline{\overline{U}}} & K_{\overline{F}_Y \overline{U}} & K_{\overline{F}_Y \overline{\overline{V}}} & K_{\overline{F}_Y \overline{V}} & K_{\overline{F}_Y \overline{\overline{W}}} & K_{\overline{F}_Y \overline{W}} \\
K_{\overline{\overline{F}}_Z \overline{\overline{U}}} & K_{\overline{\overline{F}}_Z \overline{U}} & K_{\overline{\overline{F}}_Z \overline{\overline{V}}} & K_{\overline{\overline{F}}_Z \overline{V}} & K_{\overline{\overline{F}}_Z \overline{\overline{W}}} & K_{\overline{\overline{F}}_Z \overline{W}} \\
K_{\overline{F}_Z \overline{\overline{U}}} & K_{\overline{F}_Z \overline{U}} & K_{\overline{F}_Z \overline{\overline{V}}} & K_{\overline{F}_Z \overline{V}} & K_{\overline{F}_Z \overline{\overline{W}}} & K_{\overline{F}_Z \overline{W}}
\end{bmatrix}
\begin{Bmatrix}
\overline{\overline{U}} \\ \overline{U} \\ \overline{\overline{V}} \\ \overline{V} \\ \overline{\overline{W}} \\ \overline{W}
\end{Bmatrix}
$$

$$\Downarrow$$

$$
\begin{Bmatrix}
\overline{F}_X \\ \overline{F}_Y \\ \overline{F}_Z
\end{Bmatrix}
=
\begin{bmatrix}
K_{\overline{F}_X \overline{\overline{U}}} & K_{\overline{F}_X \overline{U}} & K_{\overline{F}_X \overline{\overline{V}}} & K_{\overline{F}_X \overline{V}} & K_{\overline{F}_X \overline{\overline{W}}} & K_{\overline{F}_X \overline{W}} \\
K_{\overline{F}_Y \overline{\overline{U}}} & K_{\overline{F}_Y \overline{U}} & K_{\overline{F}_Y \overline{\overline{V}}} & K_{\overline{F}_Y \overline{V}} & K_{\overline{F}_Y \overline{\overline{W}}} & K_{\overline{F}_Y \overline{W}} \\
K_{\overline{F}_Z \overline{\overline{U}}} & K_{\overline{F}_Z \overline{U}} & K_{\overline{F}_Z \overline{\overline{V}}} & K_{\overline{F}_Z \overline{V}} & K_{\overline{F}_Z \overline{\overline{W}}} & K_{\overline{F}_Z \overline{W}}
\end{bmatrix}
\begin{Bmatrix}
\overline{\overline{U}} \\ \overline{U} \\ \overline{\overline{V}} \\ \overline{V} \\ \overline{\overline{W}} \\ \overline{W}
\end{Bmatrix}
$$

変位が既知のものは移項する

$$
\begin{Bmatrix} \bar{F}_X \\ \bar{F}_Y \\ \bar{F}_Z \end{Bmatrix}
=
\begin{bmatrix}
K_{\bar{F}_X\bar{\bar{U}}} & K_{\bar{F}_X\bar{U}} & K_{\bar{F}_X\bar{\bar{V}}} & K_{\bar{F}_X\bar{V}} & K_{\bar{F}_X\bar{\bar{W}}} & K_{\bar{F}_X\bar{W}} \\
K_{\bar{F}_Y\bar{\bar{U}}} & K_{\bar{F}_Y\bar{U}} & K_{\bar{F}_Y\bar{\bar{V}}} & K_{\bar{F}_Y\bar{V}} & K_{\bar{F}_Y\bar{\bar{W}}} & K_{\bar{F}_Y\bar{W}} \\
K_{\bar{F}_Z\bar{\bar{U}}} & K_{\bar{F}_Z\bar{U}} & K_{\bar{F}_Z\bar{\bar{V}}} & K_{\bar{F}_Z\bar{V}} & K_{\bar{F}_Z\bar{\bar{W}}} & K_{\bar{F}_Z\bar{W}}
\end{bmatrix}
\begin{Bmatrix} \bar{\bar{U}} \\ \bar{U} \\ \bar{\bar{V}} \\ \bar{V} \\ \bar{\bar{W}} \\ \bar{W} \end{Bmatrix}
$$

$$
\begin{Bmatrix}
\bar{F}_X + K_{\bar{F}_X\bar{\bar{U}}}\bar{\bar{U}} + K_{\bar{F}_X\bar{\bar{V}}}\bar{\bar{V}} + K_{\bar{F}_X\bar{\bar{W}}}\bar{\bar{W}} \\
\bar{F}_Y + K_{\bar{F}_Y\bar{\bar{U}}}\bar{\bar{U}} + K_{\bar{F}_Y\bar{\bar{V}}}\bar{\bar{V}} + K_{\bar{F}_Y\bar{\bar{W}}}\bar{\bar{W}} \\
\bar{F}_Z + K_{\bar{F}_Z\bar{\bar{U}}}\bar{\bar{U}} + K_{\bar{F}_Z\bar{\bar{V}}}\bar{\bar{V}} + K_{\bar{F}_Z\bar{\bar{W}}}\bar{\bar{W}}
\end{Bmatrix}
=
\begin{bmatrix}
K_{\bar{F}_X\bar{U}} & K_{\bar{F}_X\bar{V}} & K_{\bar{F}_X\bar{W}} \\
K_{\bar{F}_Y\bar{U}} & K_{\bar{F}_Y\bar{V}} & K_{\bar{F}_Y\bar{W}} \\
K_{\bar{F}_Z\bar{U}} & K_{\bar{F}_Z\bar{V}} & K_{\bar{F}_Z\bar{W}}
\end{bmatrix}
\begin{Bmatrix} \bar{U} \\ \bar{V} \\ \bar{W} \end{Bmatrix}
$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} F_{11}F_{11} & F_{12}F_{12} & F_{13}F_{13} & 2F_{11}F_{12} & 2F_{12}F_{13} & 2F_{11}F_{13} \\ F_{21}F_{21} & F_{22}F_{22} & F_{23}F_{23} & 2F_{21}F_{22} & 2F_{22}F_{23} & 2F_{21}F_{23} \\ F_{31}F_{31} & F_{32}F_{32} & F_{33}F_{33} & 2F_{31}F_{32} & 2F_{32}F_{33} & 2F_{31}F_{33} \\ F_{11}F_{21} & F_{12}F_{22} & F_{13}F_{23} & F_{11}F_{22} + F_{12}F_{21} & F_{12}F_{23} + F_{13}F_{22} & F_{11}F_{23} + F_{13}F_{21} \\ F_{21}F_{31} & F_{22}F_{32} & F_{23}F_{33} & F_{21}F_{32} + F_{22}F_{31} & F_{22}F_{33} + F_{23}F_{32} & F_{21}F_{33} + F_{23}F_{31} \\ F_{31}F_{11} & F_{32}F_{12} & F_{33}F_{13} & F_{31}F_{12} + F_{32}F_{11} & F_{32}F_{13} + F_{33}F_{12} & F_{31}F_{13} + F_{33}F_{11} \end{bmatrix} \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \\ S_{12} \\ S_{23} \\ S_{31} \end{Bmatrix}$$