FEM：02

たった3例題で有限要素法を理解する

3

① 移流方程式 （第1回）

② 梁のたわみ

③ 拡散方程式

拡散方程式

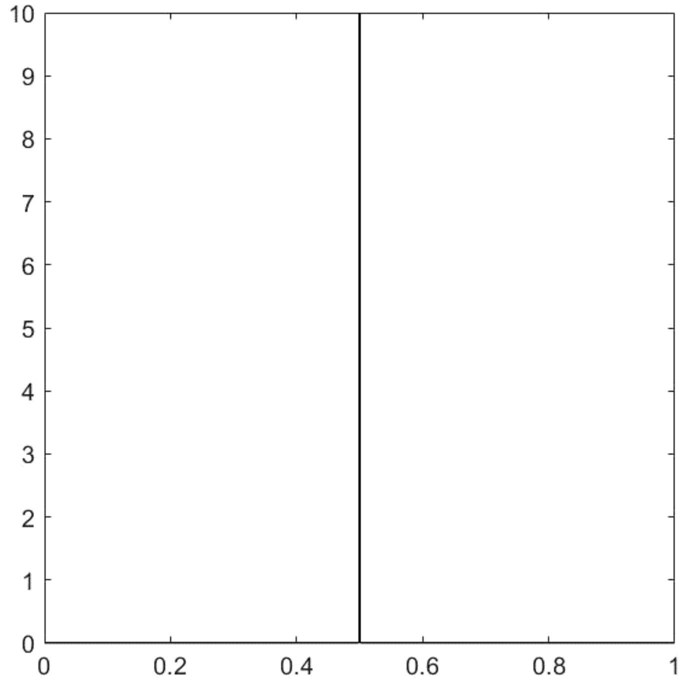
拡散方程式

物理量

モノが広がる様子を
表す微分方程式

拡散方程式

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$



微分方程式:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

解析解:

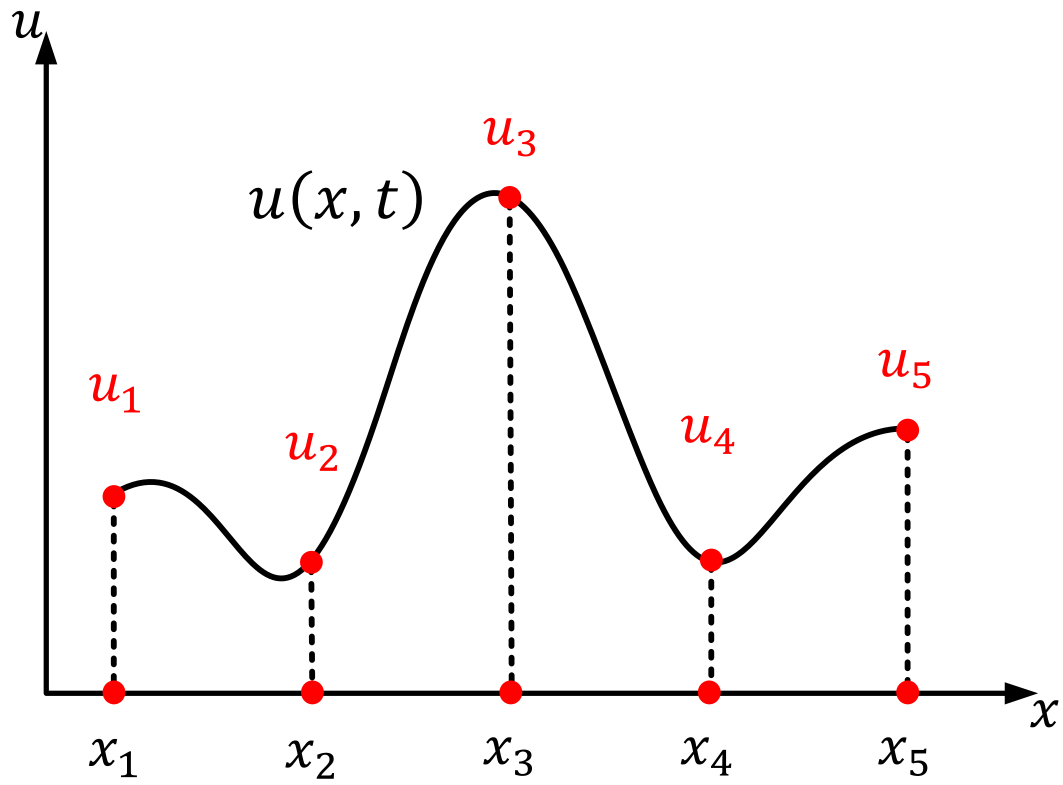
$$u = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(-\frac{x^2}{4\kappa t}\right)$$

数値解を求める
とは何か？

解析解：$u(x,t)$

数値解：$\boldsymbol{u}(t) = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$

時刻tで変化



解析解：$u(x,t)$

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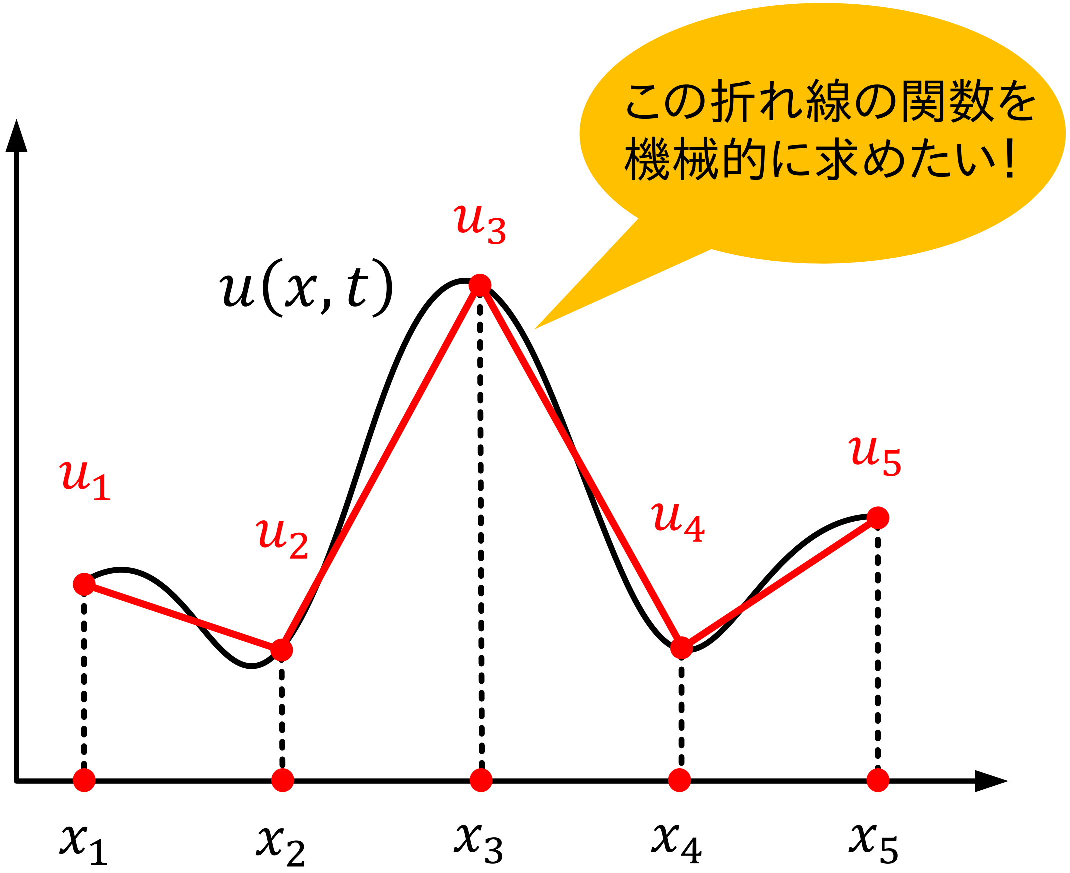
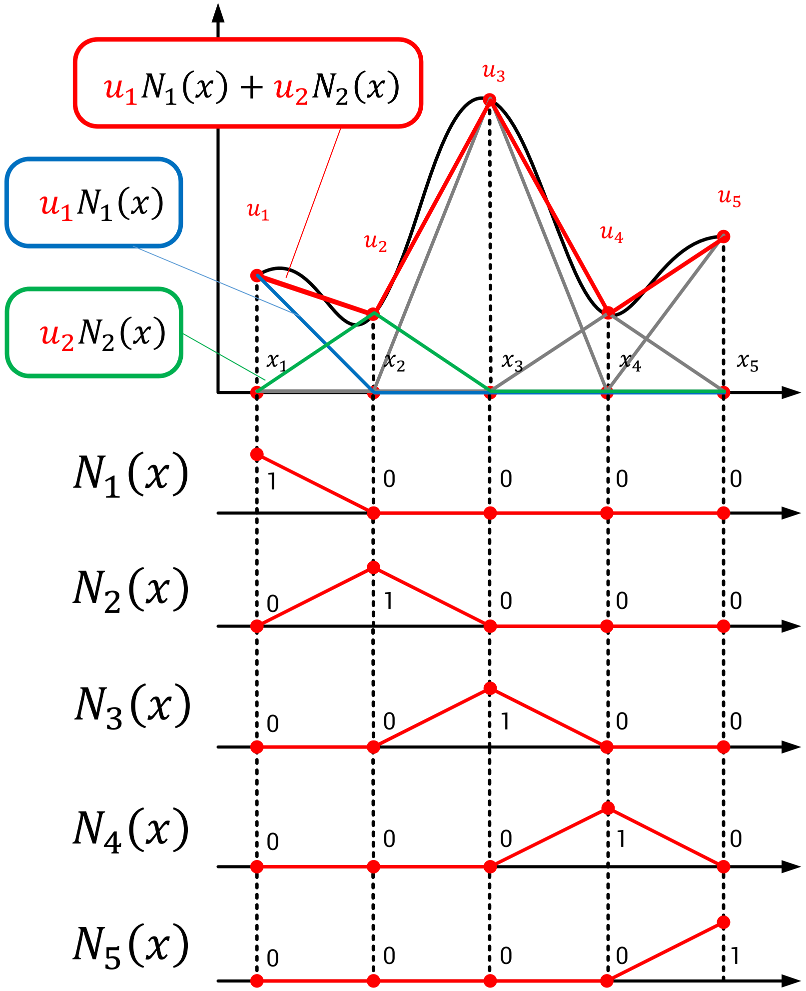
時刻tで変化



内挿



$$u_1N_1(x)+u_2N_2(x)+u_3N_3(x)+u_4N_4(x)+u_5N_5(x)$$

$$=\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{Bmatrix}\cdot\begin{Bmatrix}N_1(x)\\N_2(x)\\N_3(x)\\N_4(x)\\N_5(x)\end{Bmatrix}$$

内積で表す!

$$u(x,t)=\boldsymbol{u}(t)\cdot\boldsymbol{N}(x)$$

$$=\boldsymbol{u}^{\mathrm{T}}\boldsymbol{N}=\boldsymbol{N}^{\mathrm{T}}\boldsymbol{u}$$

$$[u_1\ \cdots\ u_2]\begin{Bmatrix}N_1\\\vdots\\N_2\end{Bmatrix}\qquad[N_1\ \cdots\ N_2]\begin{Bmatrix}u_1\\\vdots\\u_2\end{Bmatrix}$$

スカラー!

ベクトルとベクトル
の内積もスカラー

$$u(x,t) = \boldsymbol{u}(t) \cdot \boldsymbol{N}(x)$$

未知!

既知の
基底関数

Rの平均がゼロ

$$\int_{x_1}^{x_5} R \, \mathrm{d}x = 0$$

↓

満たすことが出来ない

重み付き残差wRの平均がゼロ

$$\int_{x_1}^{x_5} wR \, \mathrm{d}x = 0$$

（重み付き残差法）

任意の重み関数:$w(x)$

⬇

$\boldsymbol{u}$が一意に定まる

重み付き残差法✕Galerkin法

① 微分方程式：

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \rightarrow \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = 0$$

② 残差式にする

$$\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} = R$$

③ 重み付き残差の平均がゼロ

$$\int_0^L wR\mathrm{d}x = 0 \rightarrow \int_0^L w\left(\frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2}\right)\mathrm{d}x = 0$$

④ 解uと重みwの数値解を代入

$$\begin{cases} u = \boldsymbol{u}\cdot\boldsymbol{N} = \boldsymbol{N}^T\boldsymbol{u} \\ w = \boldsymbol{w}\cdot\boldsymbol{N} = \boldsymbol{w}^T\boldsymbol{N} \end{cases} \rightarrow \boldsymbol{w}^T\int_0^L \boldsymbol{N}\left(\boldsymbol{N}^T\frac{\partial \boldsymbol{u}}{\partial t} - \kappa \frac{\partial^2 \boldsymbol{N}^T}{\partial x^2}\boldsymbol{u}\right)\mathrm{d}x = 0$$

重みwに同じ基底関数を
用いる方法をGalerkin法という

折れ線の基底関数は
1次式なので2階微分は**ゼロ**！

弱形式化

弱形式化

部分積分を利用して、**2階微分を1階微分に変換**する操作のこと

$$\int_0^L w\left(\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2}\right)\mathrm{d}x = 0$$

2階微分が邪魔

$$\int_0^L w\frac{\partial u}{\partial t}\mathrm{d}x - \kappa\int_0^L w\frac{\partial^2 u}{\partial x^2}\mathrm{d}x = 0$$

$$= \kappa\int_0^L w\left(\frac{\partial u}{\partial x}\right)'\mathrm{d}x$$

部分積分の利用

$$= \left[w\frac{\partial u}{\partial x}\right]_0^L - \kappa\int_0^L \frac{\partial w}{\partial x}\frac{\partial u}{\partial x}\mathrm{d}x$$

$= w(0) = w(L) = 0$

重み付き残差法×Galerkin法

③ 重み付き残差の平均がゼロ

$$\int_0^L wR\mathrm{d}x = 0 \;\rightarrow\; \int_0^L w\left(\frac{\partial u}{\partial t} - \kappa\frac{\partial^2 u}{\partial x^2}\right)\mathrm{d}x = 0$$

④ 弱形式

$$\int_0^L w\frac{\partial u}{\partial t}\mathrm{d}x + \kappa\int_0^L \frac{\partial w}{\partial x}\frac{\partial u}{\partial x}\mathrm{d}x = 0$$

⑤ 解uと重みwの数値解を代入

$$\begin{cases} u = \boldsymbol{u}\cdot\boldsymbol{N} = \boldsymbol{N}^T\boldsymbol{u} \\ w = \boldsymbol{w}\cdot\boldsymbol{N} = \boldsymbol{w}^T\boldsymbol{N} \end{cases} \;\rightarrow\; \boldsymbol{w}^T\int_0^L \boldsymbol{N}\boldsymbol{N}^T\frac{\partial \boldsymbol{u}}{\partial t}\mathrm{d}x + \kappa\boldsymbol{w}^T\int_0^L \frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial \boldsymbol{N}^T}{\partial x}\boldsymbol{u}\mathrm{d}x = 0$$

$$\dot{\boldsymbol{u}} = \frac{\partial \boldsymbol{u}}{\partial t}$$

$$\boldsymbol{w}^T\left(\left[\int_0^L \boldsymbol{N}\boldsymbol{N}^T\mathrm{d}x\right]\left\{\dot{\boldsymbol{u}}\right\} + \kappa\left[\int_0^L \frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial \boldsymbol{N}^T}{\partial x}\mathrm{d}x\right]\left\{\boldsymbol{u}\right\}\right) = 0$$

$$\boldsymbol{w}^T\left(\left[\int_0^L \boldsymbol{N}\boldsymbol{N}^T \mathrm{d}x\right]\left\{\dot{\boldsymbol{u}}\right\} + \kappa\left[\int_0^L \frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial \boldsymbol{N}^T}{\partial x}\mathrm{d}x\right]\left\{\boldsymbol{u}\right\}\right) = 0$$

任意の$\boldsymbol{w}$について
常に等号成立（イコールゼロ）

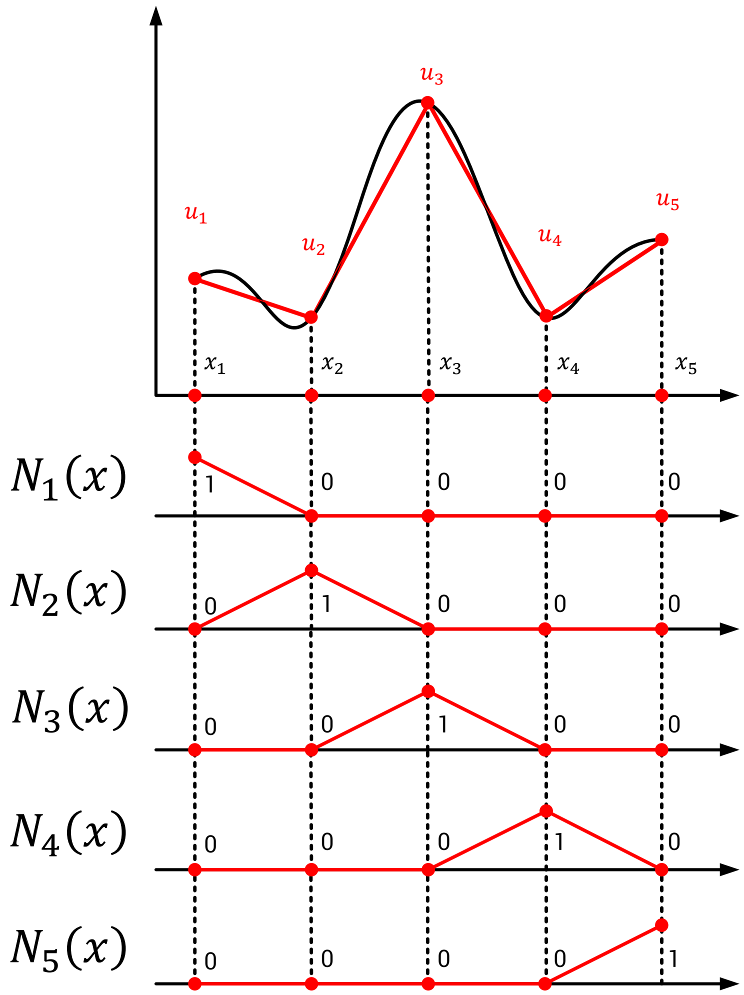
$$\left[\int_0^L \boldsymbol{N}\boldsymbol{N}^T \mathrm{d}x\right]\left\{\dot{\boldsymbol{u}}\right\} + \kappa\left[\int_0^L \frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial \boldsymbol{N}^T}{\partial x}\mathrm{d}x\right]\left\{\boldsymbol{u}\right\} = \left\{\boldsymbol{0}\right\}$$

$$\boldsymbol{N}(x) = \begin{Bmatrix} N_1(x) \\ \vdots \\ N_i(x) \\ \vdots \\ N_n(x) \end{Bmatrix}$$

i番目の節点で
山を持つ折れ線

グローバル座標系
で定義した状態

積分するタイミングで
ローカル座標系へ座標変換



重み付き残差:wRの
平均がゼロ

$$\boldsymbol{w}^{\mathrm{T}}\left(\left[\int_{x_1}^{x_5} \boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}\mathrm{d}x\right]\dot{\boldsymbol{u}} + \left[\int_{x_1}^{x_5} \frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial x}\mathrm{d}x\right]\boldsymbol{u}\right) = 0$$

$$\boldsymbol{N}\boldsymbol{N}^{\mathrm{T}} = \begin{Bmatrix} N_1 \\ \vdots \\ N_5 \end{Bmatrix}\begin{bmatrix} N_1 & \cdots & N_5 \end{bmatrix} = \begin{bmatrix} & & \\ & N_iN_j & \\ & & \end{bmatrix}$$

$$\frac{\partial \boldsymbol{N}}{\partial x}\frac{\partial \boldsymbol{N}^{\mathrm{T}}}{\partial x} = \begin{Bmatrix} \partial N_1/\partial \mathrm{x} \\ \vdots \\ \partial N_5/\partial \mathrm{x} \end{Bmatrix}\begin{bmatrix} \frac{\partial N_1}{\partial x} & \cdots & \frac{\partial N_5}{\partial x} \end{bmatrix} = \begin{bmatrix} & & \\ & \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} & \\ & & \end{bmatrix}$$

$$\boldsymbol{w}^{\mathrm{T}}(\mathbf{M}\dot{\boldsymbol{u}} + \mathbf{K}\boldsymbol{u}) = 0$$

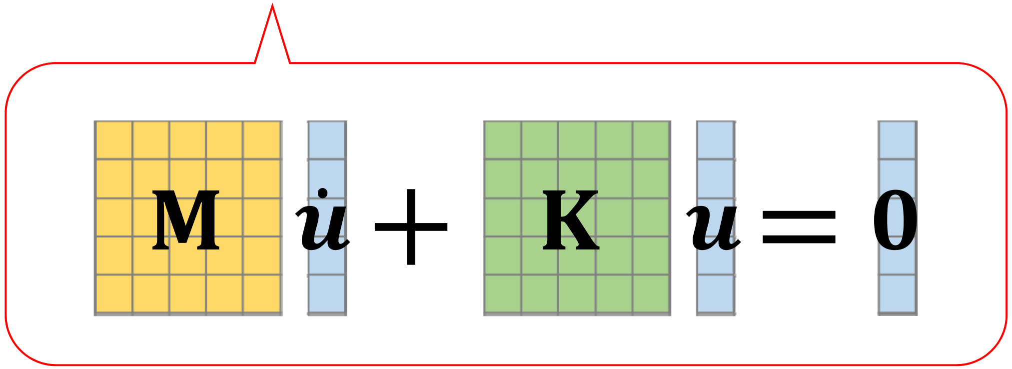
任意の$\boldsymbol{w}$に対して
等号が常に成立

$$\mathbf{M}\dot{\boldsymbol{u}} + \mathbf{K}\boldsymbol{u} = \mathbf{0}$$

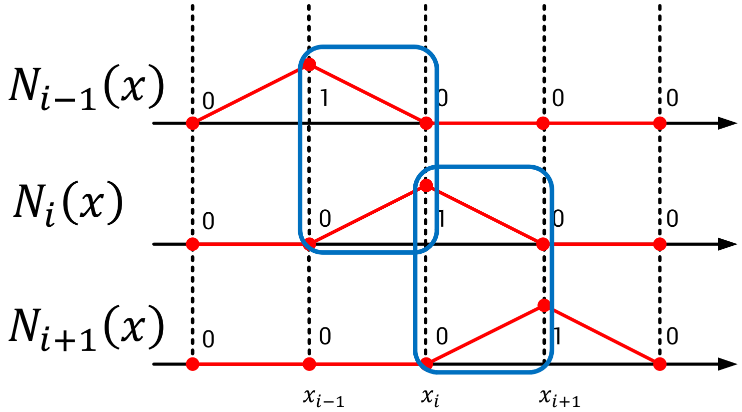
右辺は
ゼロ・ベクトル

有限要素式

$$\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{K}\boldsymbol{u}(t) = \mathbf{0}$$



$$\mathbf{M} = \left[\int_0^L \boldsymbol{N}\boldsymbol{N}^T \mathrm{d}x \right]$$



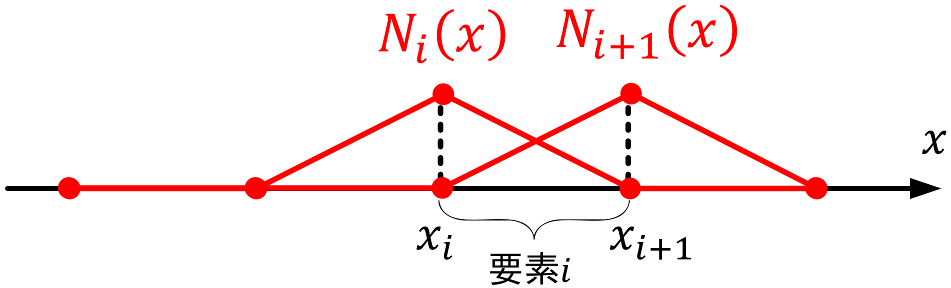
$$M_{ij} = \left[\int_0^L \boldsymbol{N}\boldsymbol{N}^T \mathrm{d}x \right]_{ij}$$

$$= \left[\int_0^L \left\{ N_i(x) \right\} \left\{ N_j(x) \right\}^{\mathrm{T}} \mathrm{d}x \right]_{ij}$$

$$= \int_0^L N_i(x) N_j(x) \mathrm{d}x$$

$$= \int_{x_1}^{x_2} N_i(x) N_j(x) \mathrm{d}x + \int_{x_2}^{x_3} N_i(x) N_j(x) \mathrm{d}x + \int_{x_3}^{x_4} N_i(x) N_j(x) \mathrm{d}x + \cdots$$

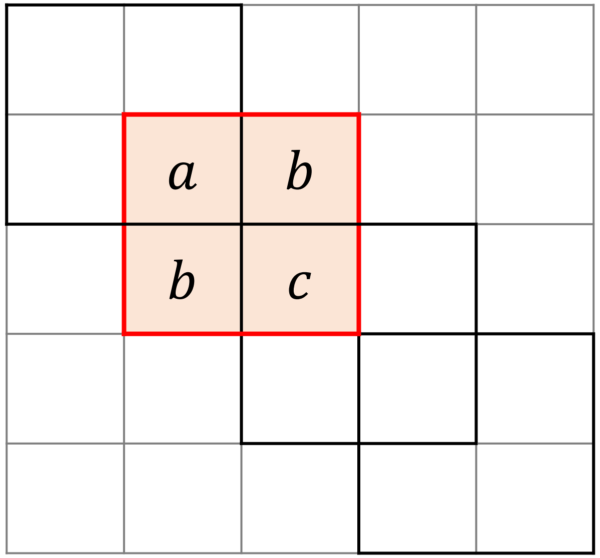
$$\mathbf{M} = \begin{bmatrix} \int_{x_1}^{x_2} N_1 N_1 \mathrm{d}x & \int_{x_1}^{x_2} N_1 N_2 \mathrm{d}x & & & \\ \int_{x_1}^{x_2} N_2 N_1 \mathrm{d}x & \int_{x_1}^{x_2} N_2 N_2 \mathrm{d}x + \int_{x_2}^{x_3} N_2 N_2 \mathrm{d}x & \int_{x_2}^{x_3} N_2 N_3 \mathrm{d}x & & \\ & \int_{x_2}^{x_3} N_3 N_2 \mathrm{d}x & \int_{x_2}^{x_3} N_3 N_3 \mathrm{d}x + \int_{x_3}^{x_4} N_3 N_3 \mathrm{d}x & \int_{x_3}^{x_4} N_3 N_4 \mathrm{d}x & \\ & & \int_{x_3}^{x_4} N_4 N_3 \mathrm{d}x & \int_{x_3}^{x_4} N_4 N_4 \mathrm{d}x + \int_{x_4}^{x_5} N_4 N_4 \mathrm{d}x & \int_{x_4}^{x_5} N_4 N_5 \mathrm{d}x \\ & & & \int_{x_4}^{x_5} N_5 N_4 \mathrm{d}x & \int_{x_4}^{x_5} N_5 N_5 \mathrm{d}x \end{bmatrix}$$

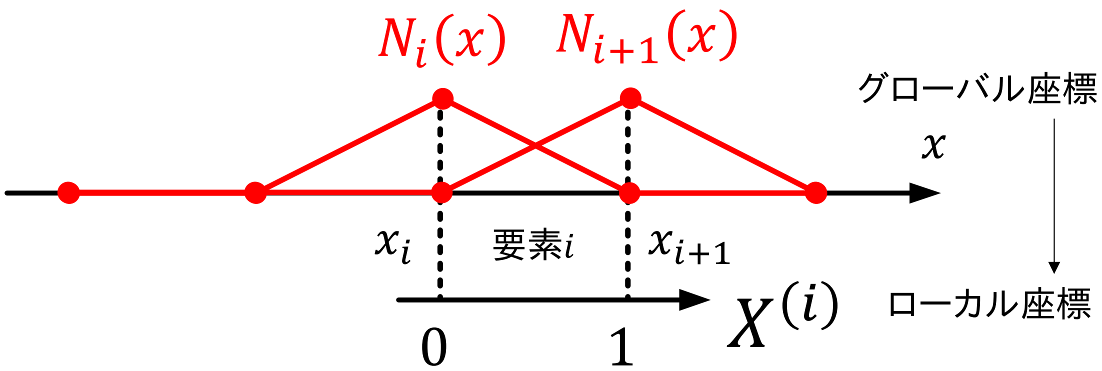


$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$



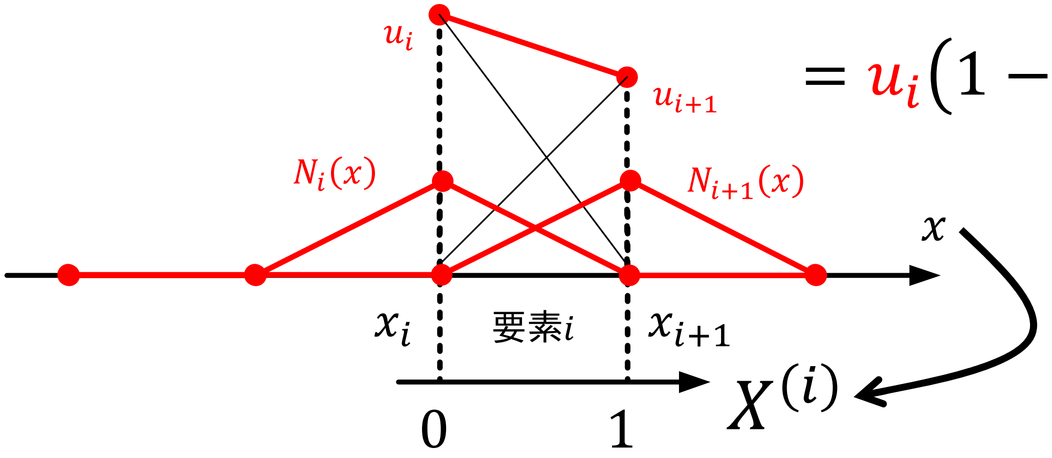


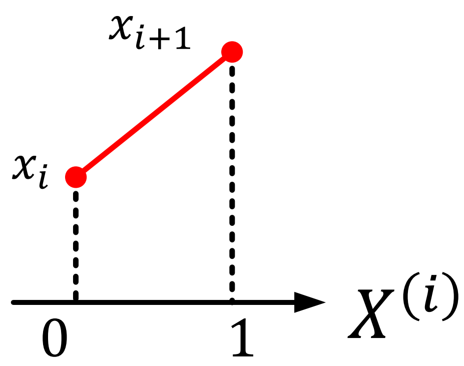
$$N_i(x) \Rightarrow N_1^{(i)}\left(X^{(i)}\right) = 1 - X^{(i)}$$

$$N_{i+1}(x) \Rightarrow N_2^{(i)}\left(X^{(i)}\right) = X^{(i)}$$

$$u_iN_i(x)+u_{i+1}N_{i+1}(x) = u_iN_1^{(i)}\left(X^{(i)}\right) + u_{i+1}N_2^{(i)}\left(X^{(i)}\right)$$

$$= u_i\left(1-X^{(i)}\right) + u_{i+1}X^{(i)}$$

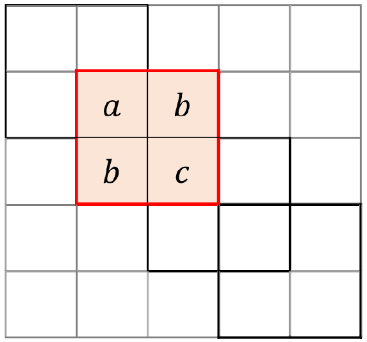




$N_i(x)$ ➡ $N_1^{(i)}\left(X^{(i)}\right) = 1 - X^{(i)}$

$N_{i+1}(x)$ ➡ $N_2^{(i)}\left(X^{(i)}\right) = X^{(i)}$

$$x = x_i\left(1-X^{(i)}\right) + x_{i+1}X^{(i)}$$



$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$

$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

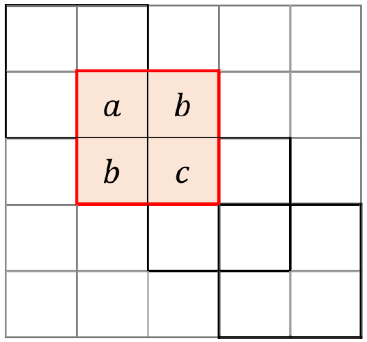
$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}X^{(i)}} \mathrm{d}X^{(i)} = \frac{\mathrm{d}}{\mathrm{d}X^{(i)}}\left(x_i N_1^{(i)}\left(X^{(i)}\right) + x_{i+1} N_2^{(i)}\left(X^{(i)}\right)\right) \mathrm{d}X^{(i)}$$

$$= \frac{\mathrm{d}}{\mathrm{d}X^{(i)}}\left(x_i\left(1 - X^{(i)}\right) + x_{i+1} X^{(i)}\right) \mathrm{d}X^{(i)} = \left(-x_i + x_{i+1}\right) \mathrm{d}X^{(i)}$$

$$= \left(x_{i+1} - x_i\right) \mathrm{d}X^{(i)} = \Delta x \, \mathrm{d}X^{(i)}$$

$$= \Delta x \int_0^1 N_1^{(i)}\left(X^{(i)}\right) N_1^{(i)}\left(X^{(i)}\right) \mathrm{d}X^{(i)}$$



$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)}$$

$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

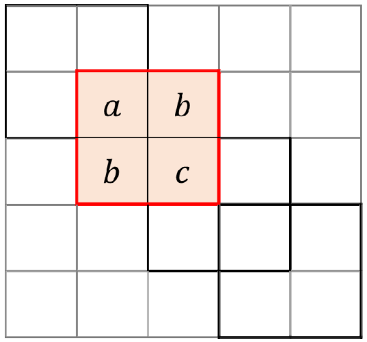
$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$

$$= \Delta x \int_0^1 N_1^{(i)}\left(X^{(i)}\right) N_1^{(i)}\left(X^{(i)}\right) \mathrm{d}X^{(i)}$$

$$= \Delta x \int_0^1 \left(1 - X^{(i)}\right)^2 \mathrm{d}X^{(i)}$$

$$= \Delta x \left[X - X^2 + \frac{1}{3} X^3 \right]_0^1 = \frac{\Delta x}{3}$$



$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right) \qquad N_{i+1}(x) = N_2^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)}$$

$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

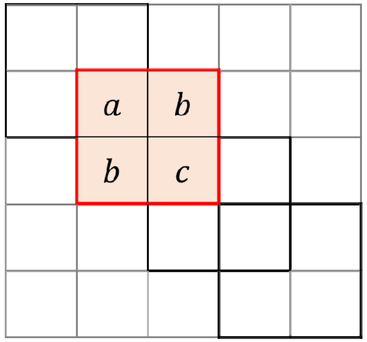
$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$

$$= \Delta x \int_0^1 N_1^{(i)}\left(X^{(i)}\right) N_2^{(i)}\left(X^{(i)}\right) \mathrm{d}X^{(i)}$$

$$= \Delta x \int_0^1 \left(1 - X^{(i)}\right) X^{(i)} \mathrm{d}X^{(i)}$$

$$= \Delta x \left[\frac{1}{2} X^2 - \frac{1}{3} X^3\right]_0^1 = \frac{\Delta x}{6}$$



$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$

$$N_{i+1}(x) = N_2^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)}$$

$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

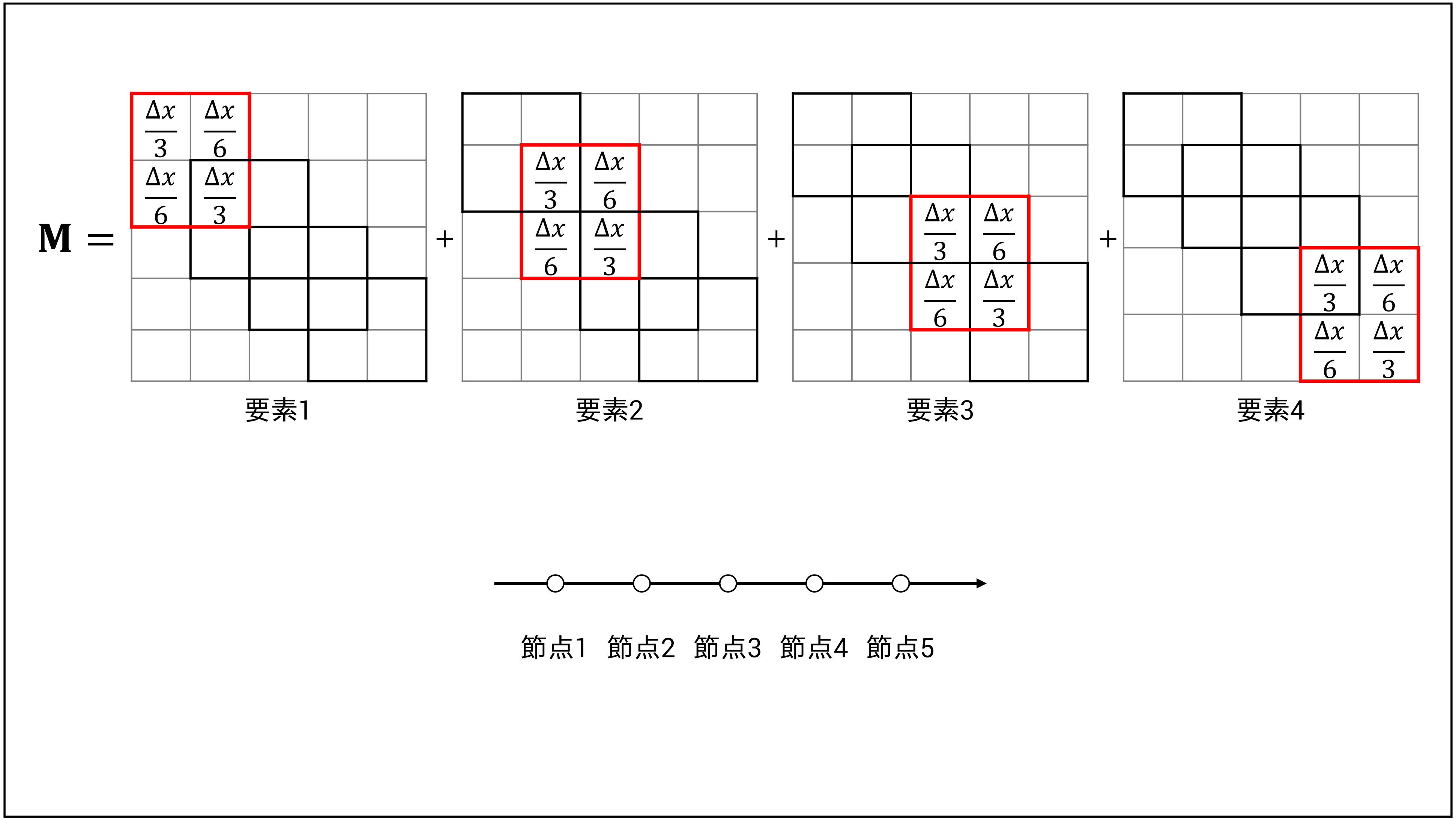
$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

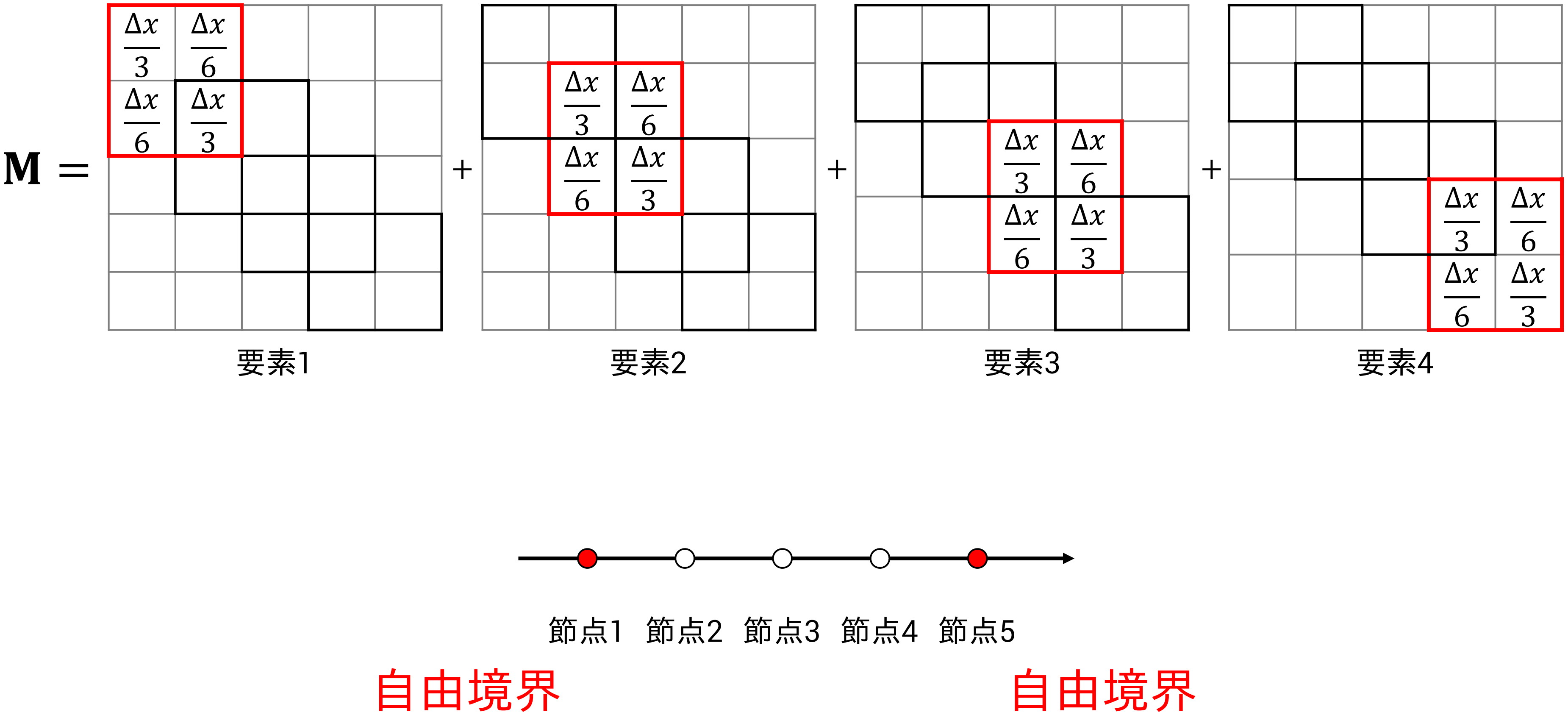
$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$

$$= \Delta x \int_0^1 N_2^{(i)}\left(X^{(i)}\right) N_2^{(i)}\left(X^{(i)}\right) \mathrm{d}X^{(i)}$$

$$= \Delta x \int_0^1 \left(X^{(i)}\right)^2 \mathrm{d}X^{(i)}$$

$$= \Delta x \left[\frac{1}{3} X^3\right]_0^1 = \frac{\Delta x}{3}$$


M =
Δx/3 Δx/6
Δx/6 Δx/3
+
要素1
要素2
要素3
要素4
節点1 節点2 節点3 節点4 節点5


M =
Δx/3
Δx/6
Δx/6
Δx/3
+
要素1
要素2
要素3
要素4
節点1
節点2
節点3
節点4
節点5
自由境界
自由境界

有限要素式

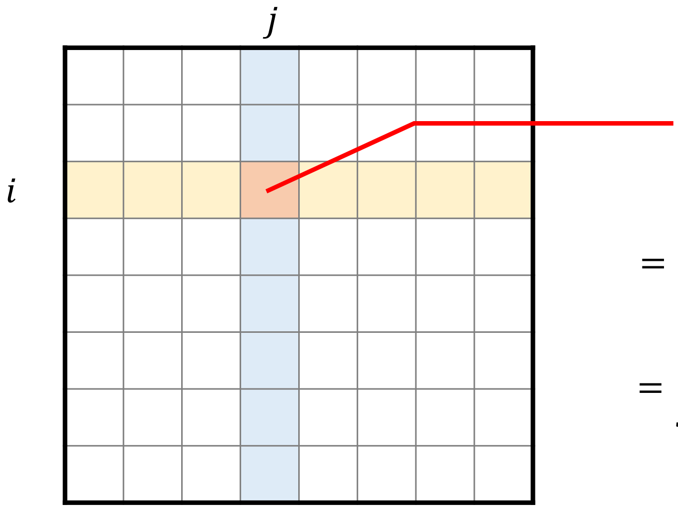
$$\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{K}\boldsymbol{u}(t) = \mathbf{0}$$

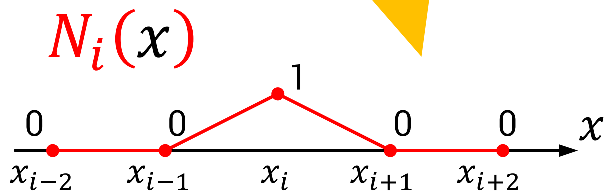
$$\begin{bmatrix} \frac{\Delta x}{3} & \frac{\Delta x}{6} & & & & & & \\ \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & & \\ & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & \\ & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & \\ & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & \\ & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & \\ & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} \\ & & & & & & \frac{\Delta x}{6} & \frac{\Delta x}{3} \end{bmatrix} \dot{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}$$

$\mathbf{K}$:剛性マトリクス

基底関数:$N_i(x)$とは、
対応する節点x_iで値が1になり、
それ以外の節点では0となる折れ線





$$\int_{x_1}^{x_5} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\mathrm{d}x$$

$$= \int_{x_1}^{x_2} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\mathrm{d}x + \int_{x_2}^{x_3} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\mathrm{d}x + \int_{x_3}^{x_4} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\mathrm{d}x + \int_{x_4}^{x_5} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\mathrm{d}x$$

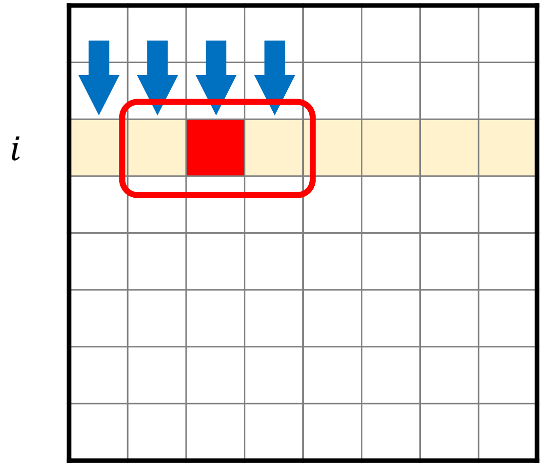
各要素での積分に分ける

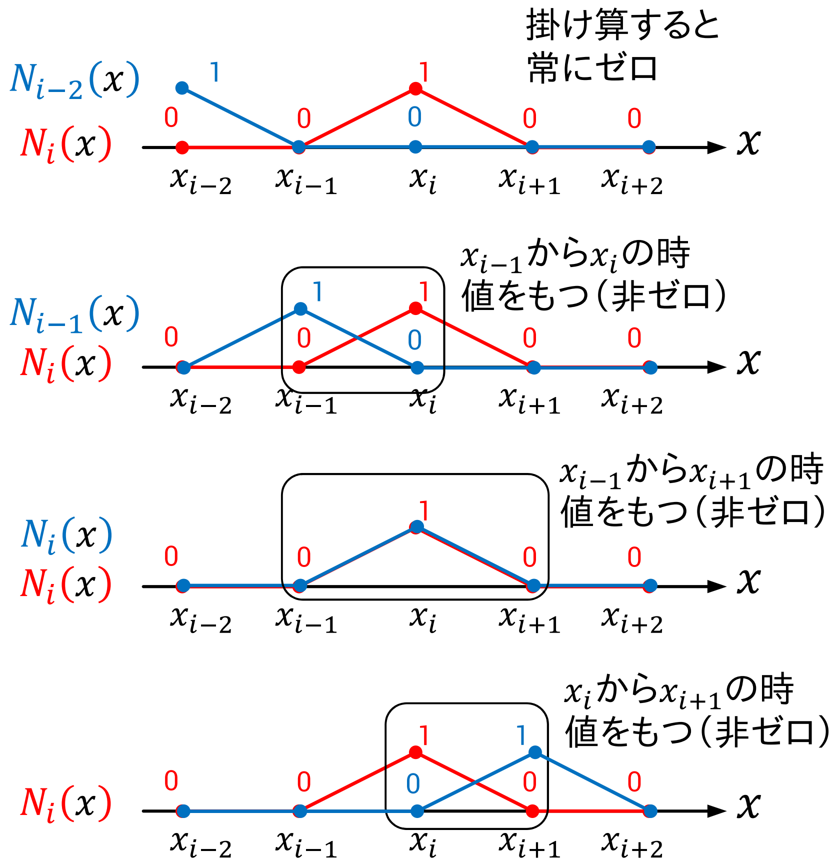
$$= \int_{x_{i-1}}^{x_i} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\mathrm{d}x + \int_{x_i}^{x_{i+1}} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\mathrm{d}x$$

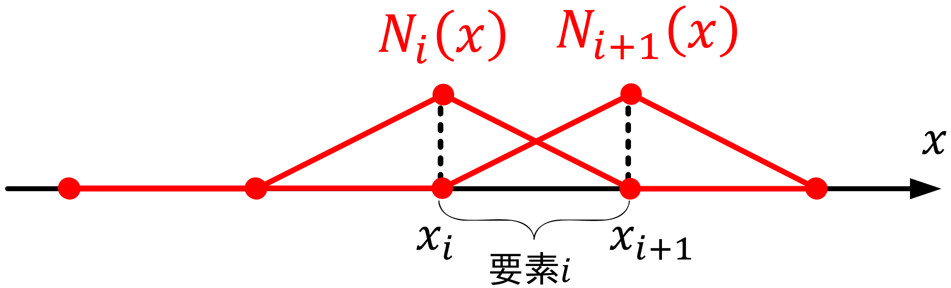
値を持ちうるのは
前後の2区間だけ

値を持ちうるのは前後の2区間だけ

$$\int_{x_1}^{x_5} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\mathrm{d}x = \int_{x_{i-1}}^{x_i} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\mathrm{d}x + \int_{x_i}^{x_{i+1}} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\mathrm{d}x$$



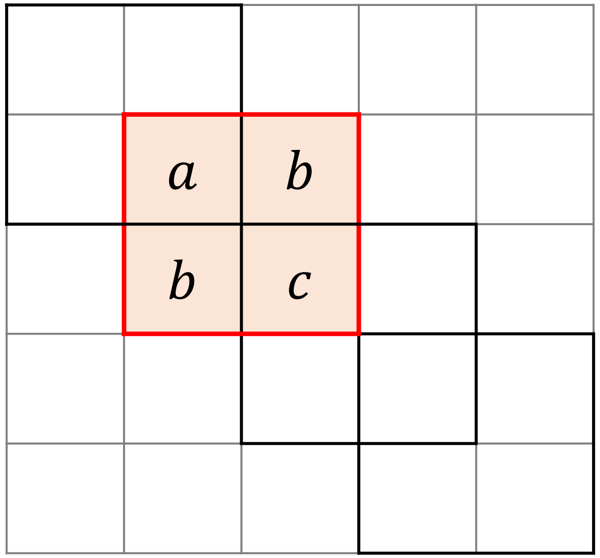


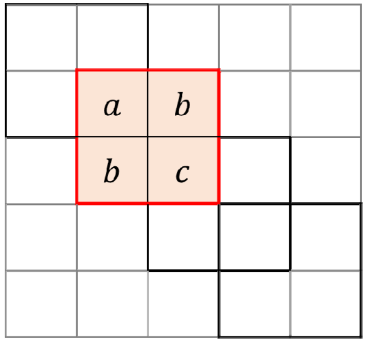


$$a = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x} \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x} \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} \frac{\partial N_{i+1}(x)}{\partial x} \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$





$$a = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x}\frac{\partial N_i(x)}{\partial x}\,\mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x}\frac{\partial N_{i+1}(x)}{\partial x}\,\mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} \frac{\partial N_{i+1}(x)}{\partial x}\frac{\partial N_i(x)}{\partial x}\,\mathrm{d}x$$

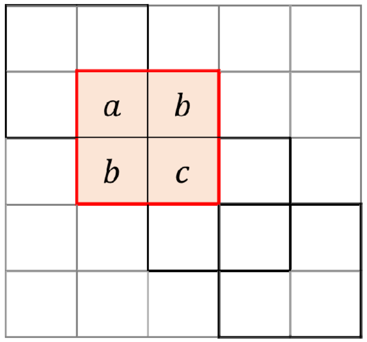
$$a = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x}\frac{\partial N_i(x)}{\partial x}\,\mathrm{d}x$$

$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x\,\mathrm{d}X^{(i)}$$

$$\frac{\partial N_i(x)}{\partial x} = \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}\frac{\partial X^{(i)}}{\partial x} = \frac{1}{\left(\frac{\partial x}{\partial X^{(i)}}\right)}\frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} = \frac{1}{\Delta x}\frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}$$

$$= \frac{1}{\Delta x}\int_0^1 \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}\frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}\,\mathrm{d}X^{(i)}$$



$$a = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x} \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x} \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} \frac{\partial N_{i+1}(x)}{\partial x} \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$a = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x} \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

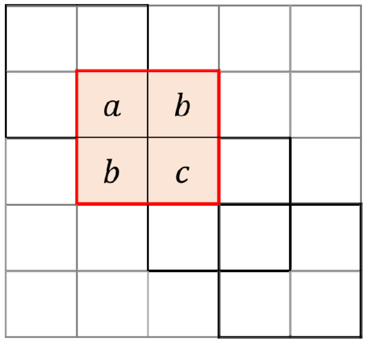
$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)} \qquad \frac{\partial N_i(x)}{\partial x} = \frac{1}{\Delta x} \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}$$

$$= \frac{1}{\Delta x} \int_0^1 \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \mathrm{d}X^{(i)}$$

$$= \frac{1}{\Delta x} \int_0^1 (-1)(-1) \mathrm{d}X^{(i)}$$

$$= \frac{1}{\Delta x} [X]_0^1 = \frac{1}{\Delta x}$$



$$a = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x} \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x} \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} \frac{\partial N_{i+1}(x)}{\partial x} \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x} \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

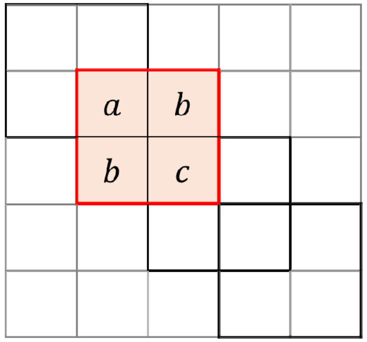
$N_i(x) = N_1^{(i)}\left(X^{(i)}\right)$ $\quad N_{i+1}(x) = N_2^{(i)}\left(X^{(i)}\right)$

$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)}$ $\quad \frac{\partial N_i(x)}{\partial x} = \frac{1}{\Delta x} \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}$

$$= \frac{1}{\Delta x} \int_0^1 \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \frac{\partial N_2^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \mathrm{d}X^{(i)}$$

$$= \frac{1}{\Delta x} \int_0^1 (-1)(1) \mathrm{d}X^{(i)}$$

$$= \frac{1}{\Delta x} \left[-X\right]_0^1 = -\frac{1}{\Delta x}$$



$$a = \int_{x_i}^{x_{i+1}} \frac{\partial N_i(x)}{\partial x} \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

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$$c = \int_{x_i}^{x_{i+1}} \frac{\partial N_{i+1}(x)}{\partial x} \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} \frac{\partial N_{i+1}(x)}{\partial x} \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$N_i(x) = N_1^{(i)}\left(X^{(i)}\right)$ $\quad N_{i+1}(x) = N_2^{(i)}\left(X^{(i)}\right)$

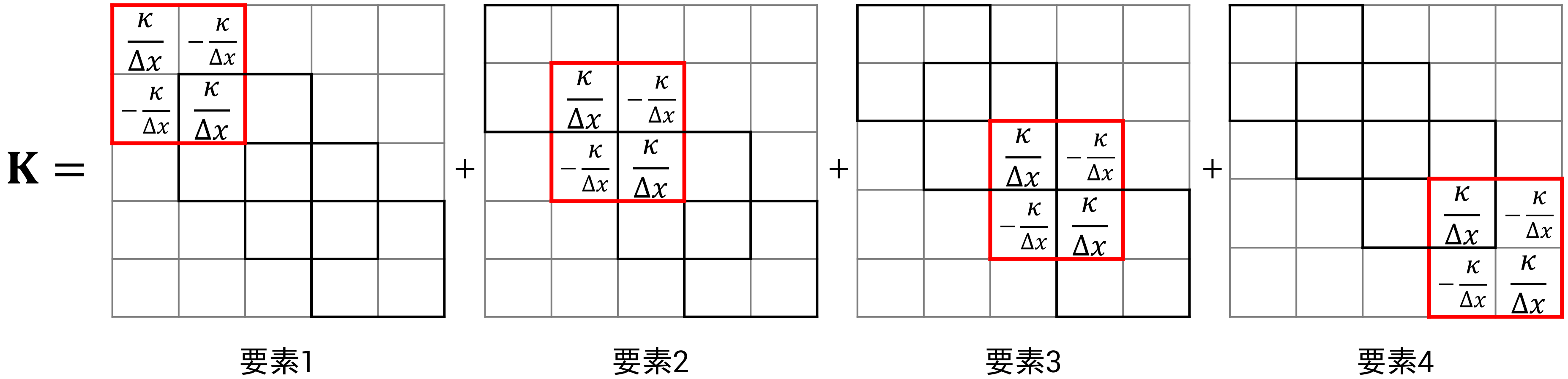
$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)}$ $\quad \dfrac{\partial N_i(x)}{\partial x} = \dfrac{1}{\Delta x} \dfrac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}$

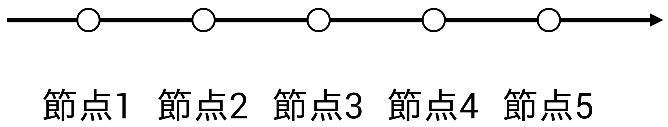
$$= \frac{1}{\Delta x} \int_0^1 \frac{\partial N_2^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \frac{\partial N_2^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \mathrm{d}X^{(i)}$$

$$= \frac{1}{\Delta x} \int_0^1 (1)(1) \mathrm{d}X^{(i)}$$

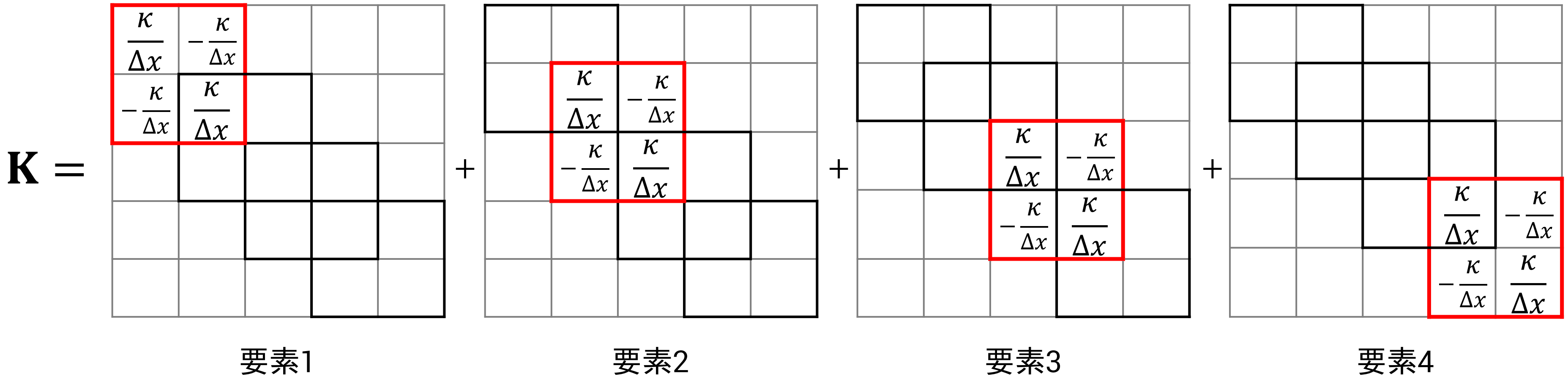
$$= \frac{1}{\Delta x} [X]_0^1 \quad = \frac{1}{\Delta x}$$

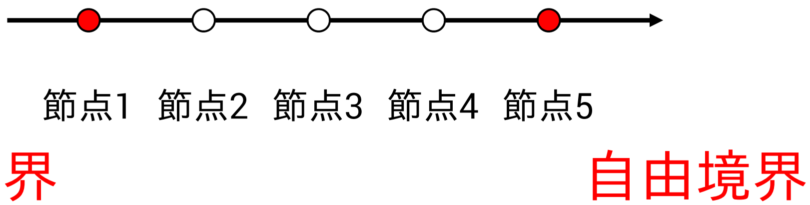
係数κを忘れないこと





係数κを忘れないこと





有限要素式

$$\mathbf{M}\dot{\boldsymbol{u}}(t)+\mathbf{K}\boldsymbol{u}(t)=\mathbf{0}$$

$$
\begin{pmatrix}
\frac{\Delta x}{3} & \frac{\Delta x}{6} & & & & & & \\
\frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & & \\
 & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & \\
 & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & \\
 & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & \\
 & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & \\
 & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} \\
 & & & & & & \frac{\Delta x}{6} & \frac{\Delta x}{3}
\end{pmatrix}
\dot{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}}
\qquad
\begin{pmatrix}
\frac{\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & & & & & & \\
-\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & & & & & \\
 & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & & & & \\
 & & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & & & \\
 & & & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & & \\
 & & & & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & \\
 & & & & & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} \\
 & & & & & & -\frac{\kappa}{\Delta x} & \frac{\kappa}{\Delta x}
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}
$$

前進差分法

$$\dot{\boldsymbol{u}}(t) = \frac{\boldsymbol{u}(t + \Delta t) - \boldsymbol{u}(t)}{\Delta t}$$

↓

計算不可は小さいが
不安定な場合が多い

後退差分法

$$\dot{\boldsymbol{u}}(t) = \frac{\boldsymbol{u}(t) - \boldsymbol{u}(t - \Delta t)}{\Delta t}$$

↓

計算不可は大きいが
安定な場合が多い

前進差分法で

計算負荷を小さくする方法

||

逆行列計算を無くす

前進差分法

$$\dot{\boldsymbol{u}}(t) = \frac{\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)}{\Delta t}$$

代入

有限要素式

$$\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{K}\boldsymbol{u}(t) = \mathbf{0}$$

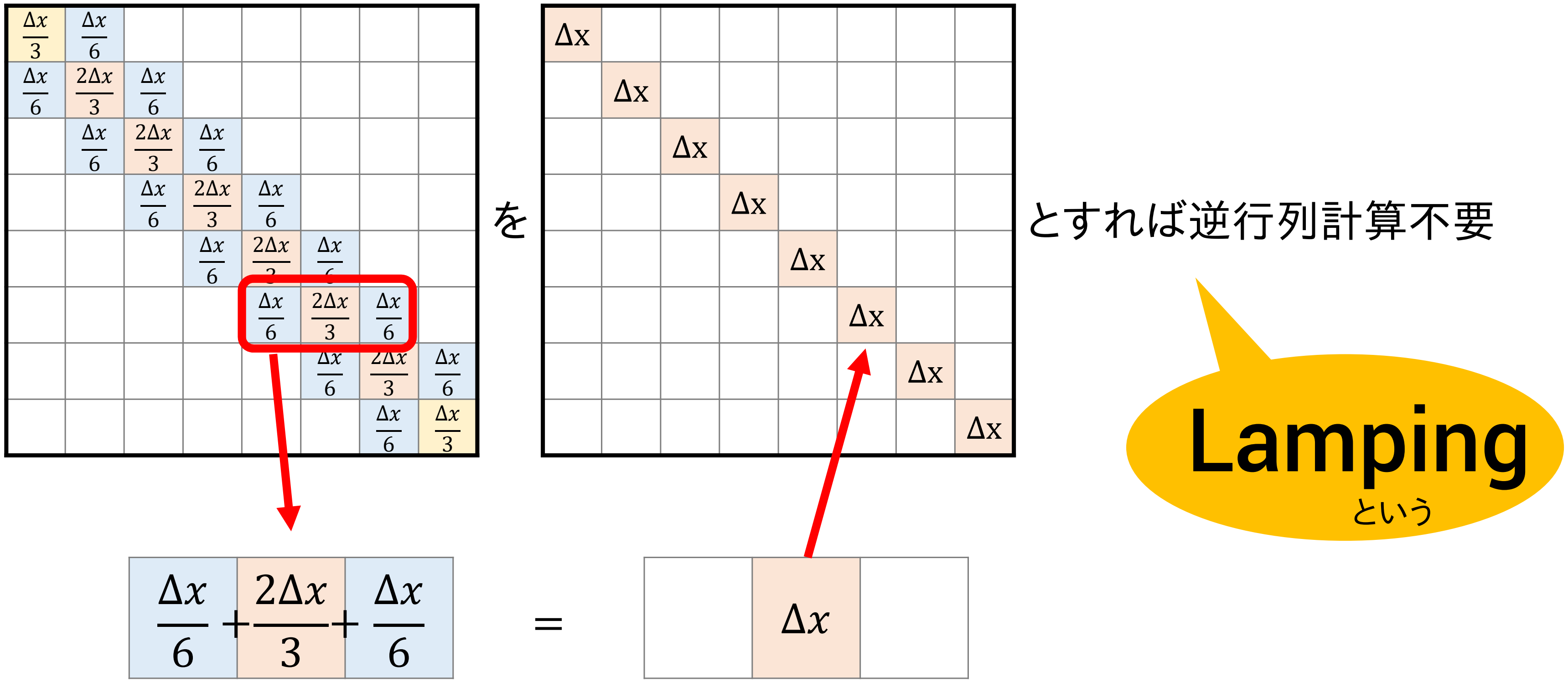
$$\mathbf{M}\left(\frac{\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)}{\Delta t}\right) + \mathbf{K}\boldsymbol{u}(t) = \mathbf{0}$$

$$\mathbf{M}\boldsymbol{u}(t+\Delta t) - \mathbf{M}\boldsymbol{u}(t) + \Delta t\mathbf{K}\boldsymbol{u}(t) = \mathbf{0}$$

$$\mathbf{M}\boldsymbol{u}(t+\Delta t) = [\mathbf{M} - \Delta t\mathbf{K}]\boldsymbol{u}(t)$$

$$\boldsymbol{u}(t+\Delta t) = \mathbf{M}^{-1}[\mathbf{M} - \Delta t\mathbf{K}]\boldsymbol{u}(t)$$

$$\boldsymbol{u}(t+\Delta t) = \mathbf{M}^{-1}[\mathbf{M} - \Delta t\mathbf{K}]\boldsymbol{u}(t)$$



$$\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{K}\boldsymbol{u}(t) = \mathbf{0}$$

前進差分法

$$\dot{\boldsymbol{u}}(t) = \frac{\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)}{\Delta t}$$

Lamping

$$\boldsymbol{u}(t+\Delta t) = \left[\mathbf{E} - \frac{\Delta t}{\Delta x}\mathbf{K}\right]\boldsymbol{u}(t)$$

後退差分法

$$\dot{\boldsymbol{u}}(t) = \frac{\boldsymbol{u}(t) - \boldsymbol{u}(t-\Delta t)}{\Delta t}$$

$$\dot{\boldsymbol{u}}(t+\Delta t) = \frac{\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)}{\Delta t}$$

$$\mathbf{M}\dot{\boldsymbol{u}}(t+\Delta t) + \mathbf{K}\boldsymbol{u}(t+\Delta t) = \mathbf{0}$$

$$\mathbf{M}\frac{\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)}{\Delta t} + \mathbf{K}\boldsymbol{u}(t+\Delta t) = \mathbf{0}$$

$$\mathbf{M}\boldsymbol{u}(t+\Delta t) - \mathbf{M}\boldsymbol{u}(t) + \Delta t\mathbf{K}\boldsymbol{u}(t+\Delta t) = \mathbf{0}$$

$$[\mathbf{M} + \Delta t\mathbf{K}]\boldsymbol{u}(t+\Delta t) = \mathbf{M}\boldsymbol{u}(t)$$

$$\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{K}\boldsymbol{u}(t) = \mathbf{0}$$

前進差分法

$$\dot{\boldsymbol{u}}(t) = \frac{\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)}{\Delta t}$$

Lamping

$$\boldsymbol{u}(t+\Delta t) = \left[\mathbf{E} - \frac{\Delta t}{\Delta x}\mathbf{K}\right]\boldsymbol{u}(t)$$

後退差分法

$$\dot{\boldsymbol{u}}(t) = \frac{\boldsymbol{u}(t) - \boldsymbol{u}(t-\Delta t)}{\Delta t}$$

$$\dot{\boldsymbol{u}}(t+\Delta t) = \frac{\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)}{\Delta t}$$

$$[\mathbf{M} + \Delta t\mathbf{K}]\boldsymbol{u}(t+\Delta t) = \mathbf{M}\boldsymbol{u}(t)$$

$$\boldsymbol{u}(t+\Delta t) = [\mathbf{M} + \Delta t\mathbf{K}]^{-1}\mathbf{M}\boldsymbol{u}(t)$$

Lampingしても計算負荷を
減らすことは出来ない

MATLABコーディング

uの初期値設定

```
u = [zeros(100,1); 200*ones(1,1); zeros(99,1)];
```

uの長さは要素数(=節点数)　※周期境界だから

```
n = length(u);
```

```
t = 0:dt:T
```

0 から T
まで刻みをdtで
数列を生成する

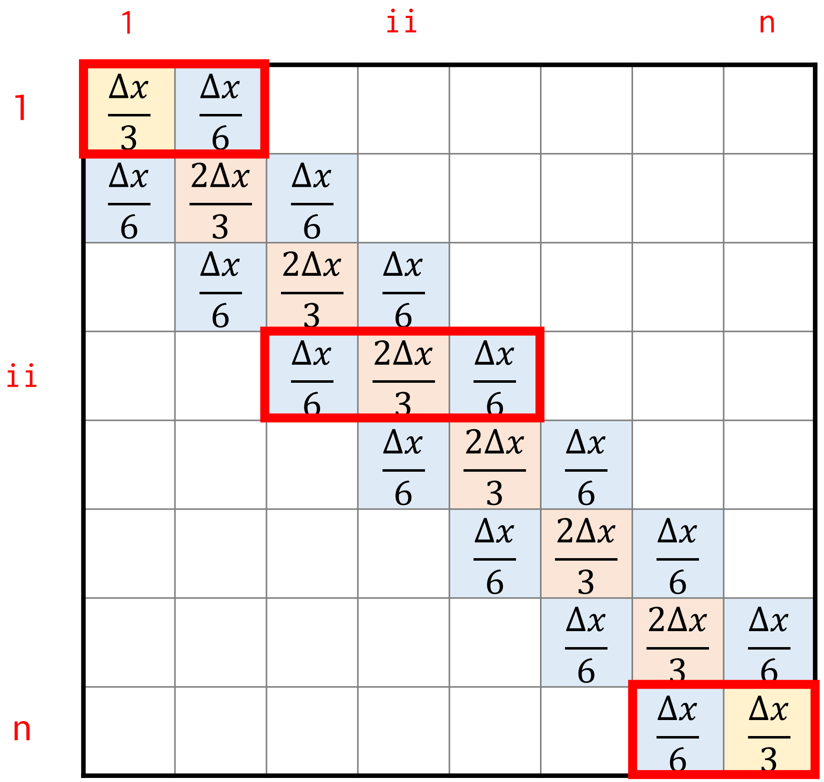
その他、パラメータを設定する

```
n = length(u); %-- num of elem
t = 0:dt:1;
T = length(t);
dx= 1/n;
x = [0:dx:(n-1)*dx];
```

```
x = 0:dx:(n-1)*dx
```

0 から (n-1)*dx
まで刻みをdxで
数列を生成する

MATLABコーディング



全ての成分がゼロのn×n行列を作成

```
M = zeros(n,n);
```

（初期化）

```
for ii=1:n
    if ii==1
        M(ii,ii)   = dx*1/3;
        M(ii,ii+1)= dx*1/6;
    elseif ii==n
        M(ii,ii-1)= dx*1/6;
        M(ii,ii)   = dx*1/3;
    else
        M(ii,ii-1)= dx*1/6;
        M(ii,ii)   = dx*2/3;
        M(ii,ii+1)= dx*1/6;
    end
end
```

MATLABコーディング

$\frac{\kappa}{\Delta x}$	$-\frac{\kappa}{\Delta x}$						
$-\frac{\kappa}{\Delta x}$	$\frac{2\kappa}{\Delta x}$	$-\frac{\kappa}{\Delta x}$					
	$-\frac{\kappa}{\Delta x}$	$\frac{2\kappa}{\Delta x}$	$-\frac{\kappa}{\Delta x}$				
		$-\frac{\kappa}{\Delta x}$	$\frac{2\kappa}{\Delta x}$	$-\frac{\kappa}{\Delta x}$			
			$-\frac{\kappa}{\Delta x}$	$\frac{2\kappa}{\Delta x}$	$-\frac{\kappa}{\Delta x}$		
				$-\frac{\kappa}{\Delta x}$	$\frac{2\kappa}{\Delta x}$	$-\frac{\kappa}{\Delta x}$	
					$-\frac{\kappa}{\Delta x}$	$\frac{2\kappa}{\Delta x}$	$-\frac{\kappa}{\Delta x}$
						$-\frac{\kappa}{\Delta x}$	$\frac{\kappa}{\Delta x}$

```
K = zeros(n,n);
for ii=1:n
    if ii==1
        K(ii,ii)   = k/dx;
        K(ii,ii+1)= -k/dx;
    elseif ii==n
        K(ii,ii-1)= -k/dx;
        K(ii,ii)   = k/dx;
    else
        K(ii,ii-1)= -k/dx;
        K(ii,ii)   = k/dx;
        K(ii,ii+1)= -k/dx;
    end
end
```

シミュレーション・コードの計算式

$$\boldsymbol{u}(t+\Delta t) = \left[\mathbf{E} - \frac{\Delta t}{\Delta x}\mathbf{K}\right]\boldsymbol{u}(t)$$

```
A1 = eye(n)-dt/dx*K;
```

初期化

```
u1 = [u zeros(n,T-1)];
```

```
for tt=1:T-1
    u1(:,tt+1)=A1*u1(:,tt);
end
```

時刻ttの時の$\boldsymbol{u}$を用いて
時刻tt+1の時の$\boldsymbol{u}$を求める

シミュレーション・コードの計算式

$$\boldsymbol{u}(t+\Delta t) = [\mathbf{M} + \Delta t \mathbf{K}]^{-1} \mathbf{M} \boldsymbol{u}(t)$$

```
A2 = (M+k*dt*K)¥M;
```

バックスラッシュ

¥は＼

左を**逆行列**にして
右に掛ける、の意

初期化

```
u2 = [u zeros(n,T-1)];
```

```
for tt=1:T-1
    u2(:,tt+1)=A2*u2(:,tt);
end
```

後退差分法では
逆行列計算が避けられない

時刻ttの時の$\boldsymbol{u}$を用いて
時刻tt+1の時の$\boldsymbol{u}$を求める

MATLABコーディング

結果のVisualization

Figure領域に
時刻ttのグラフを描画

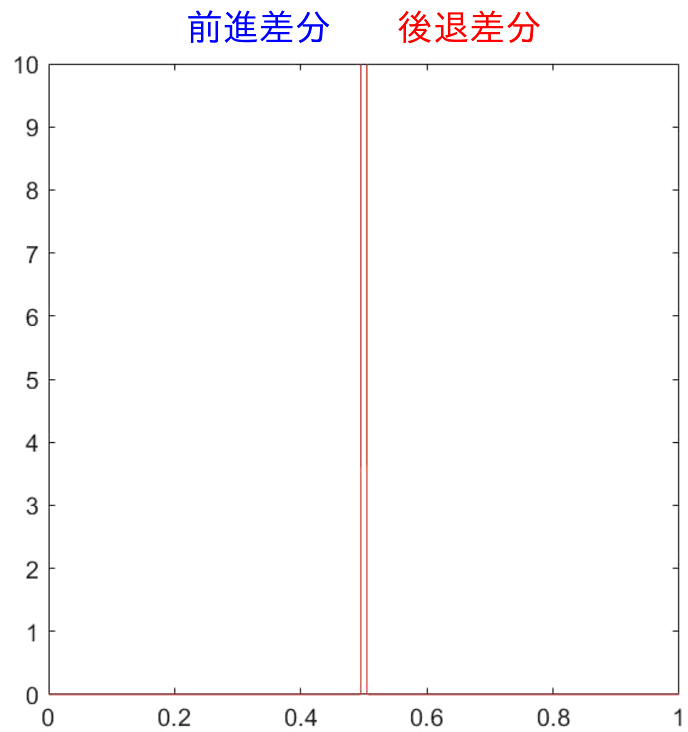
動画ファイルの
フレームに保存

時刻ttを更新

```
% open file
[y,m,d]    = ymd(datetime);
ymdstr     = [num2str(y) num2str(m,'%0.2d') num2str(d,'%0.2d')];
filename   = ['ex_04_' ymdstr '.avi'];
writerObj = VideoWriter(filename,'Motion JPEG AVI');
writerObj.FrameRate = 20; %-- flame rate (slide/sec)
open(writerObj); %-- initialization of movie object

% avi visualization
figure(1)
Fnum=200;
for tt=1:round(T/Fnum):T
    set(gcf, 'position', [100   100   480   480 ])
    set(gcf, 'color',[1 1 1])
    plot((0:dx:1)',[u1(:,tt);u1(1,tt)],'b-');hold on
    plot((0:dx:1)',[u2(:,tt);u2(1,tt)],'r-');hold off
    set(gca, 'position', [0.10  0.08  0.87  0.87])
    set(gca, 'fontname', 'Arial', 'fontsize', 12)
    xlim([0 1])
    ylim([0 10])
    frame = getframe(figure(1)); %-- get frame
    writeVideo(writerObj,frame); %-- add frame to video obj
end

close(writerObj);
```



結果は一致しているので
差が見えない

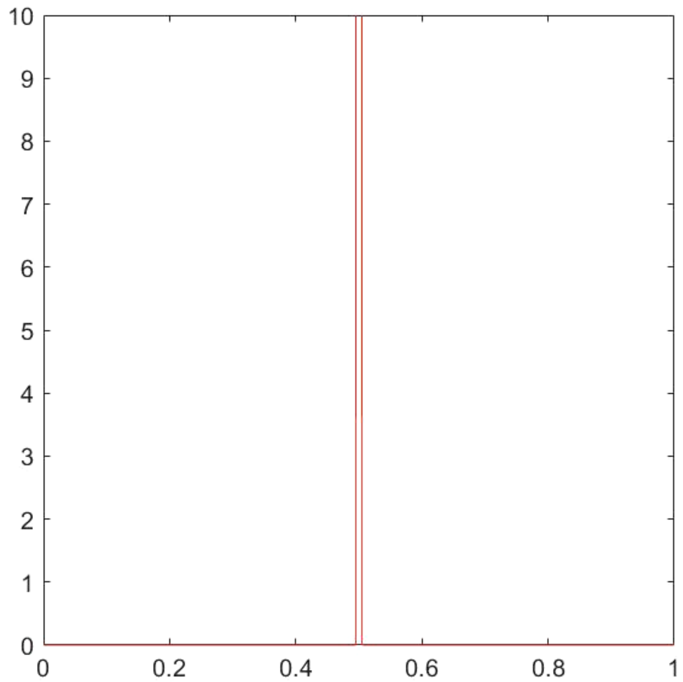
有限要素式

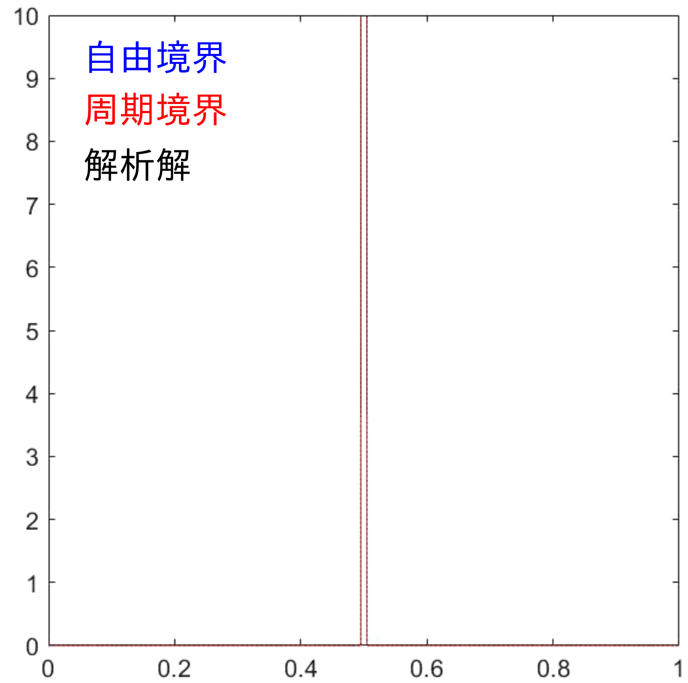
$$\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{K}\boldsymbol{u}(t) = \mathbf{0}$$

周期境界
にすることもできる

$$\begin{bmatrix} \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & & & \frac{\Delta x}{6} \\ \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & & \\ & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & \\ & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & \\ & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & \\ & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & \\ & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} \\ \frac{\Delta x}{6} & & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} \end{bmatrix} \dot{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}} \quad \begin{bmatrix} \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & & & & & & -\frac{\kappa}{\Delta x} \\ -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & & & & & \\ & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & & & & \\ & & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & & & \\ & & & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & & \\ & & & & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} & \\ & & & & & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} & -\frac{\kappa}{\Delta x} \\ -\frac{\kappa}{\Delta x} & & & & & & -\frac{\kappa}{\Delta x} & \frac{2\kappa}{\Delta x} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}$$

前進差分　後退差分




自由境界
周期境界
解析解
0 1 2 3 4 5 6 7 8 9 10
0 0.2 0.4 0.6 0.8 1