FEM：01

たった**3**例題で**有限要素法**を理解する

有限要素法

発散定理

重積分

線形変換

難解な数学

輸送定理

弱形式化

3

① 移流方程式

② 梁のたわみ

③ 拡散方程式

移流方程式

移流方程式

物理量

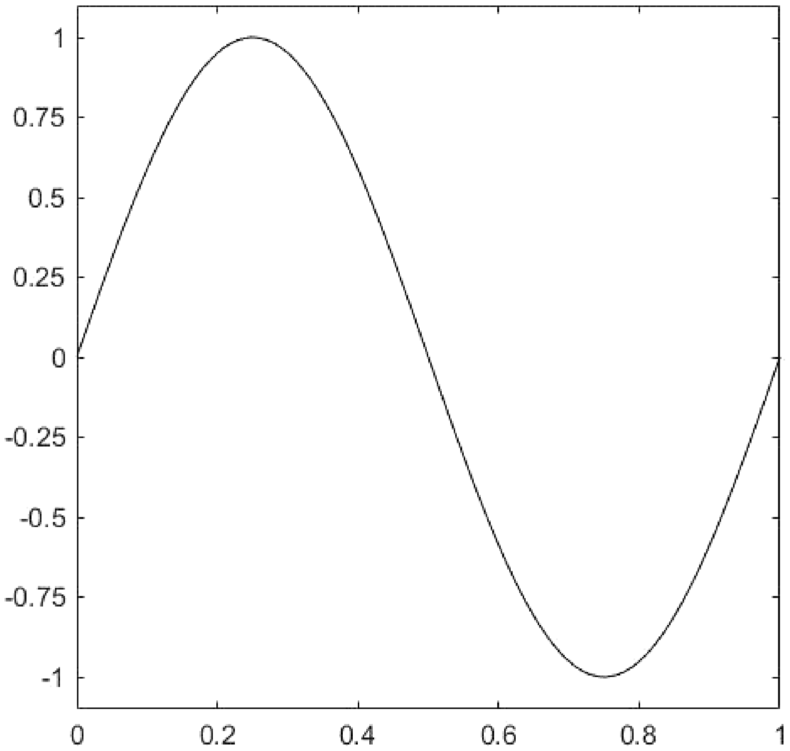
モノが流れる様子を
表す微分方程式

物理量・・・たとえば、**密度**

$$\boxed{\frac{\partial u}{\partial t}} + c\boxed{\frac{\partial u}{\partial x}} = 0$$

時間微分

空間微分



最初（$t = 0$）の波形は、$A\sin(\pi x)$

振幅: A 　　移流速度: c

$$u(x,t) = A\sin\big(\pi(x - ct)\big)$$

時刻tの波形の位置を
ctだけ戻すと、最初に戻る

$$u(x,t) = A\sin\big(\pi(x-ct)\big) \xrightarrow{\text{代入}} \boxed{\frac{\partial u}{\partial t}} + c\boxed{\frac{\partial u}{\partial x}} = 0$$

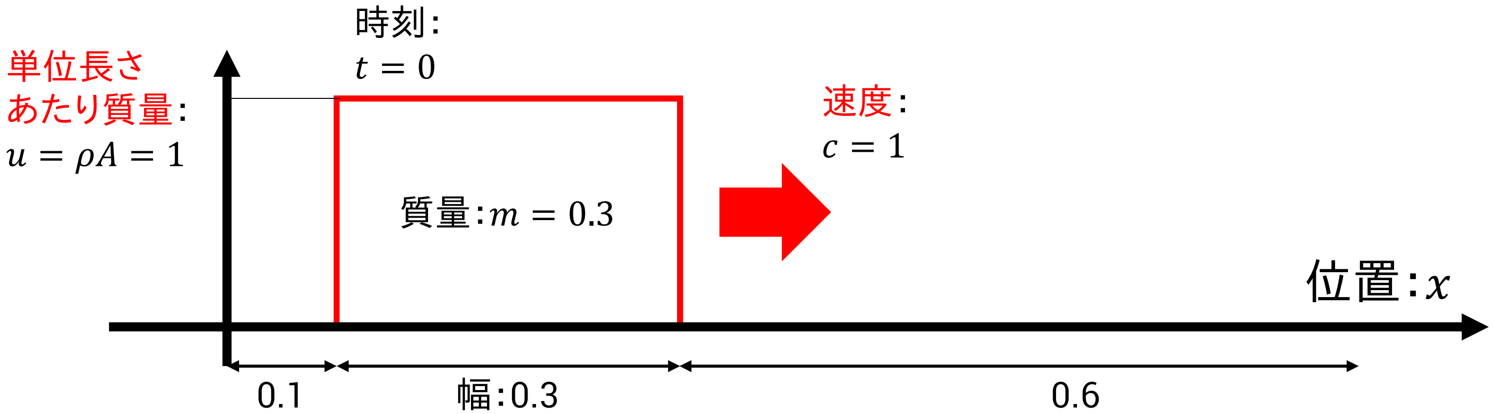
$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left\{A\sin\big(\pi(x-ct)\big)\right\}$$
$$= -\pi c\, A\cos\big(\pi(x-ct)\big)$$

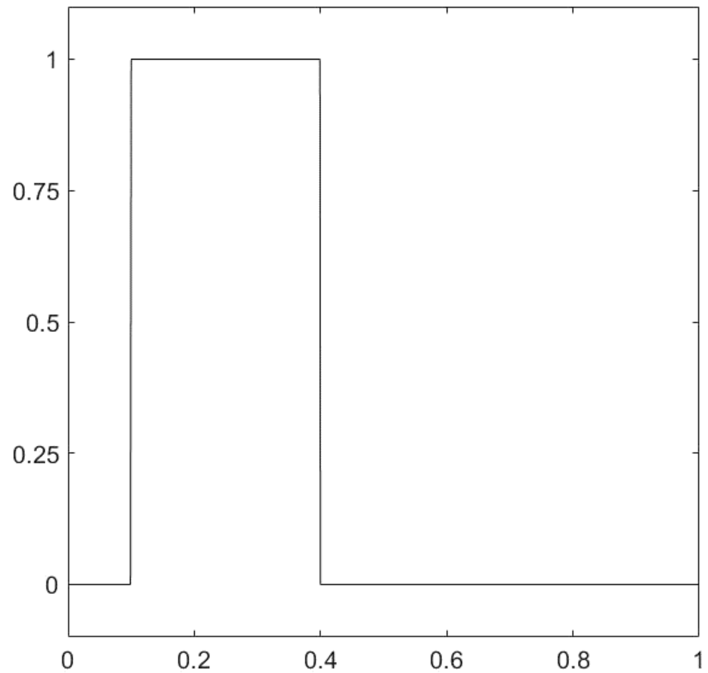
$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left\{A\sin\big(\pi(x-ct)\big)\right\}$$
$$= \pi\, A\cos\big(\pi(x-ct)\big)$$

たしかにイコール･ゼロが成り立っている!

(問題)

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$ に従う矩形波 $u(x,t)$ を求めよ。





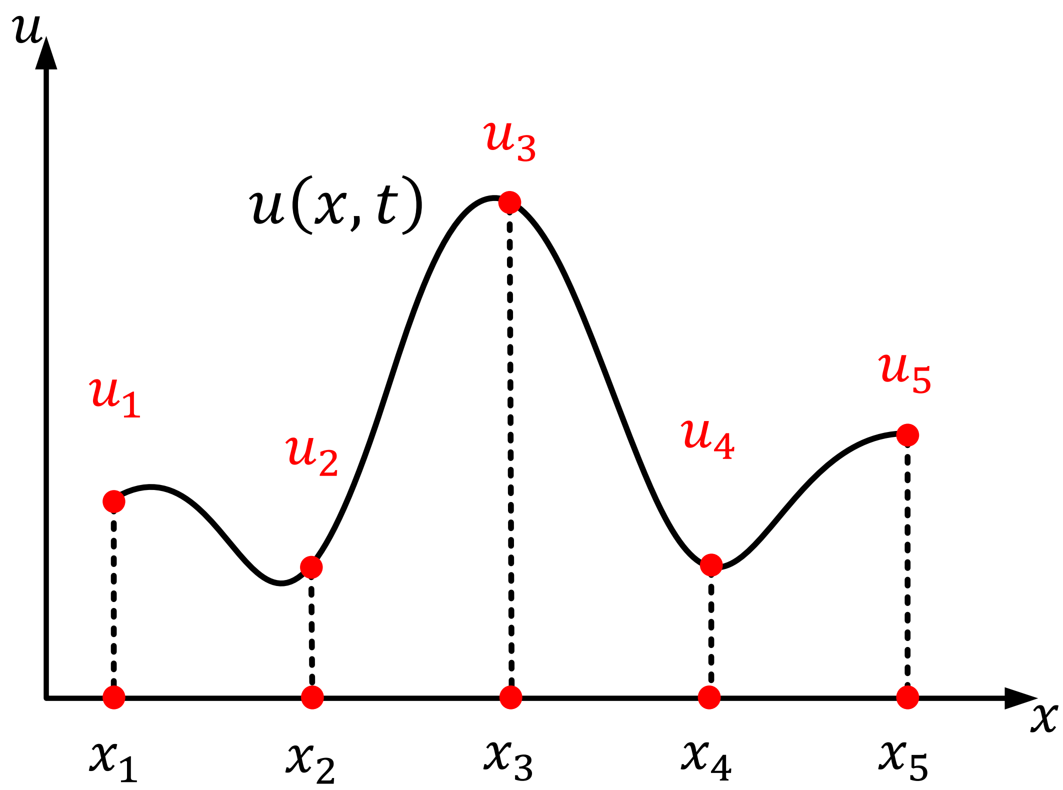
$$u(x,t)=\begin{cases}1 & (0.1+ct\le x\le 0.4+ct)\\ 0 & \text{上記以外}\end{cases}$$

数値解を求める
とは何か？

解析解：$u(x,t)$

数値解：$\boldsymbol{u}(t) = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$

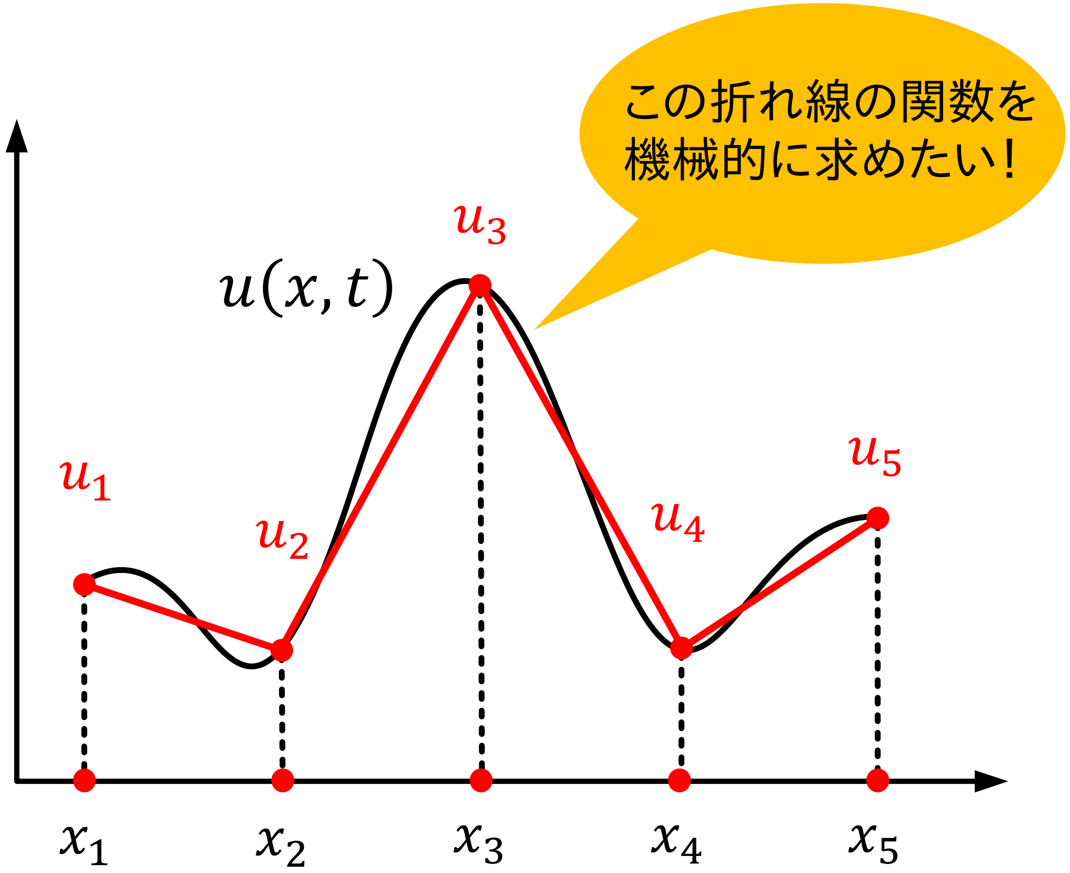
時刻tで変化



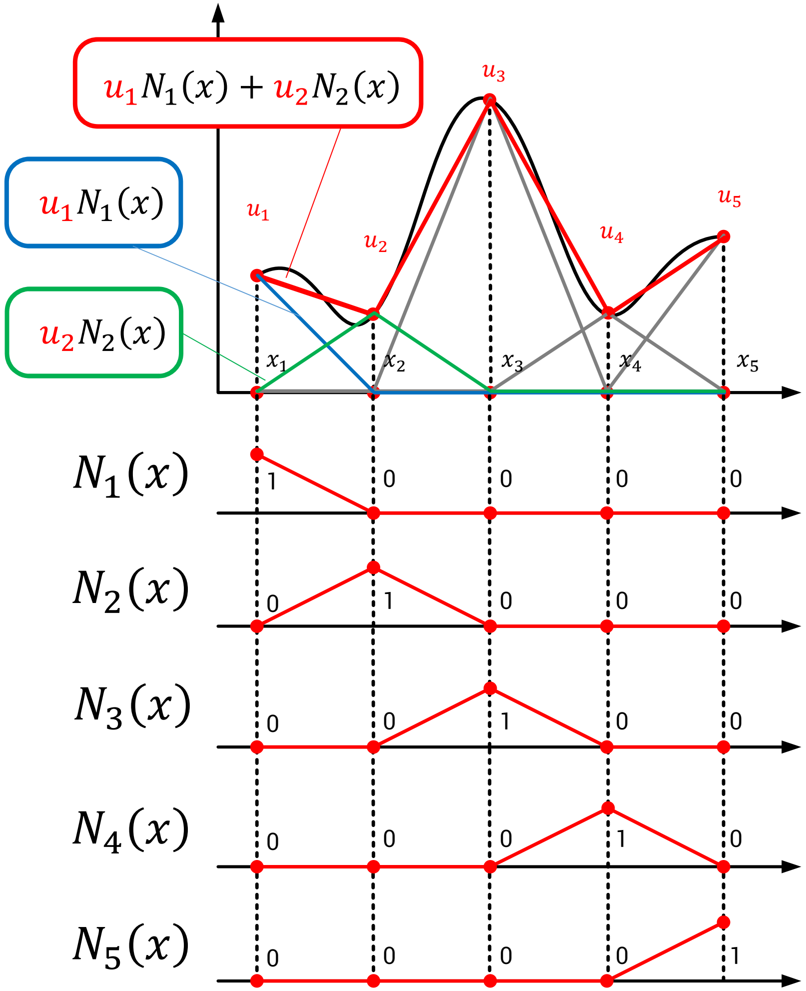
解析解：$u(x,t)$

数値解：$\boldsymbol{u}(t)=\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix}$

時刻tで変化



内挿



$$u_1N_1(x) + u_2N_2(x) + u_3N_3(x) + u_4N_4(x) + u_5N_5(x)$$

$$= \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} \cdot \begin{Bmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \\ N_4(x) \\ N_5(x) \end{Bmatrix}$$

内積で表す!

$$\boxed{u(x,t) = \boldsymbol{u}(t) \cdot \boldsymbol{N}(x)}$$

$$= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{N} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{u}$$

$$[u_1 \quad \cdots \quad u_2]\begin{Bmatrix} N_1 \\ \vdots \\ N_2 \end{Bmatrix} \qquad [N_1 \quad \cdots \quad N_2]\begin{Bmatrix} u_1 \\ \vdots \\ u_2 \end{Bmatrix}$$

スカラー!

ベクトルとベクトル
の内積もスカラー

$$u(x,t) = \boldsymbol{u}(t) \cdot \boldsymbol{N}(x)$$

未知!

既知の
基底関数

元の微分方程式

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

数値解

$$u = \boldsymbol{u} \cdot \boldsymbol{N}$$

数値解は近似解なので、元の微分方程式に代入すると、ゼロにならない

残差

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = R$$

Rの平均がゼロ

$$\int_{x_1}^{x_5} R \, \mathrm{d}x = 0$$

満たすことが出来ない

重み付き残差wRの平均がゼロ

$$\int_{x_1}^{x_5} wR\,\mathrm{d}x = 0$$

（重み付き残差法）

任意の重み関数:$w(x)$

⬇

$\boldsymbol{u}$が一意に定まる

右辺のゼロをRと
おいておく

元の微分方程式

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = R$$

代入

重み付き残差：wRの平均がゼロ

$$\int_{x_1}^{x_5} wR \,\mathrm{d}x = 0$$

重み付き残差式

$$\int_{x_1}^{x_5} w\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right)\mathrm{d}x = 0$$

$$\int_{x_1}^{x_5} w\left(\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x}\right)\mathrm{d}x = 0$$

数値解

$$u = \boldsymbol{u}\cdot\boldsymbol{N}$$

重みの数値解

$$w = \boldsymbol{w}\cdot\boldsymbol{N}$$

Galerkin法

$$\int_{x_1}^{x_5}(\boldsymbol{w}\cdot\boldsymbol{N})\left(\frac{\partial(\boldsymbol{u}\cdot\boldsymbol{N})}{\partial t} + c\,\frac{\partial(\boldsymbol{u}\cdot\boldsymbol{N})}{\partial x}\right)\mathrm{d}x = 0$$

$$\int_{x_1}^{x_5} (\boldsymbol{w}\cdot\boldsymbol{N})\left(\frac{\partial(\boldsymbol{u}\cdot\boldsymbol{N})}{\partial t} + c\frac{\partial(\boldsymbol{u}\cdot\boldsymbol{N})}{\partial x}\right)\mathrm{d}x = 0$$

$\boldsymbol{w}\cdot\boldsymbol{N} = \boldsymbol{w}^{\mathrm{T}}\boldsymbol{N}$

$\boldsymbol{u}\cdot\boldsymbol{N} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{u}$

$\boldsymbol{N}(x)$以外は
全て外に出せる

$$\int_{x_1}^{x_5} \boldsymbol{w}^{\mathrm{T}}\boldsymbol{N}\left(\boxed{\frac{\partial(\boldsymbol{N}^{\mathrm{T}}\boldsymbol{u})}{\partial t}} + c\boxed{\frac{\partial(\boldsymbol{N}^{\mathrm{T}}\boldsymbol{u})}{\partial x}}\right)\mathrm{d}x = 0$$

$$\boldsymbol{w}^{\mathrm{T}}\int_{x_1}^{x_5} \boldsymbol{N}\left(\boldsymbol{N}^{\mathrm{T}}\boxed{\frac{\partial\boldsymbol{u}}{\partial t}} + c\boxed{\frac{\partial\boldsymbol{N}^{\mathrm{T}}}{\partial x}}\boldsymbol{u}\right)\mathrm{d}x = 0$$

$$\dot{\boldsymbol{u}} = \frac{\partial\boldsymbol{u}}{\partial t}$$

$$\boldsymbol{w}^{\mathrm{T}}\left(\left[\int_{x_1}^{x_5}\boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}\mathrm{d}x\right]\boxed{\dot{\boldsymbol{u}}} + \left[\int_{x_1}^{x_5}\boldsymbol{N}\frac{\partial\boldsymbol{N}^{\mathrm{T}}}{\partial x}\mathrm{d}x\right]\boxed{\boldsymbol{u}}\right) = 0$$

重み付き残差：wRの
平均がゼロ

$$\boldsymbol{w}^{\mathrm{T}}\left(\left[\int_{x_1}^{x_5}\boxed{\boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}}\mathrm{d}x\right]\dot{\boldsymbol{u}}+\left[\int_{x_1}^{x_5}\boxed{\boldsymbol{N}\frac{\partial\boldsymbol{N}^{\mathrm{T}}}{\partial x}}\mathrm{d}x\right]\boldsymbol{u}\right)=0$$

$$\boldsymbol{N}\boldsymbol{N}^{\mathrm{T}}=\begin{Bmatrix}N_1\\ \vdots\\ N_5\end{Bmatrix}\begin{bmatrix}N_1 & \cdots & N_5\end{bmatrix}=\begin{bmatrix} & & \\ & N_iN_j & \\ & & \end{bmatrix}$$

$$\boldsymbol{N}\frac{\partial\boldsymbol{N}^{\mathrm{T}}}{\partial x}=\begin{Bmatrix}N_1\\ \vdots\\ N_5\end{Bmatrix}\begin{bmatrix}\dfrac{\partial N_1}{\partial x} & \cdots & \dfrac{\partial N_5}{\partial x}\end{bmatrix}=\begin{bmatrix} & & \\ & N_i\dfrac{\partial N_j}{\partial x} & \\ & & \end{bmatrix}$$

$$\boldsymbol{w}^{\mathrm{T}}(\mathbf{M}\dot{\boldsymbol{u}}+\mathbf{C}\boldsymbol{u})=0$$

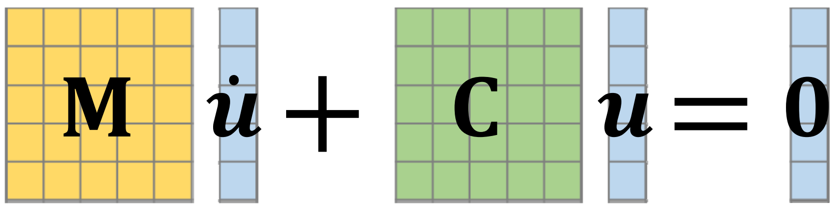
任意の$\boldsymbol{w}$に対して
等号が常に成立

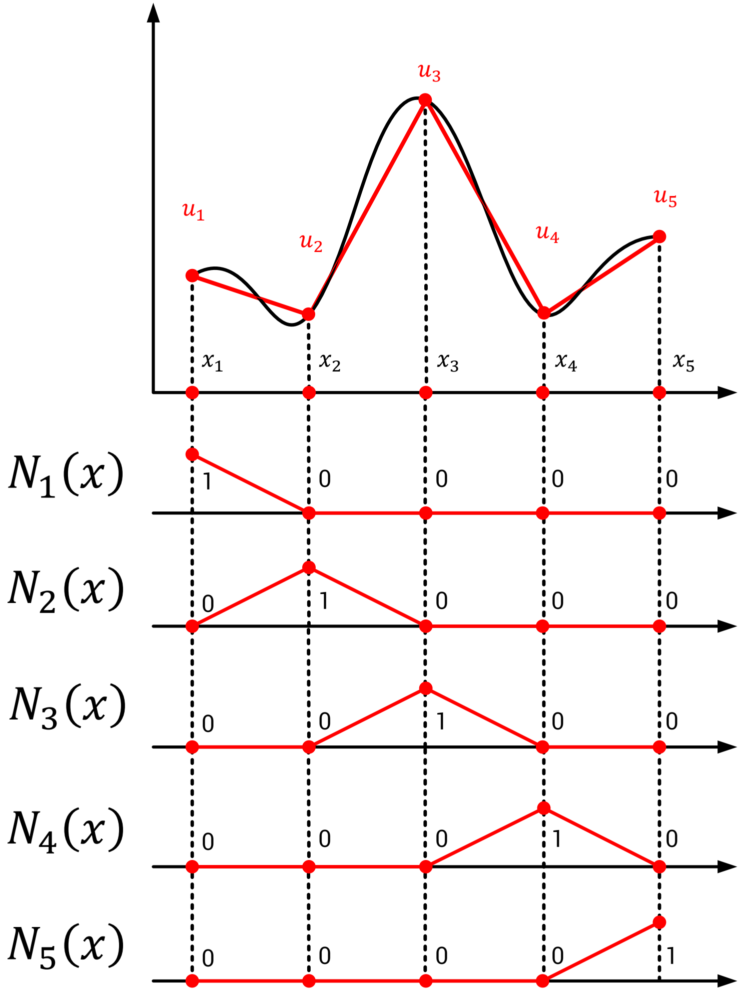
$$\mathbf{M}\dot{\boldsymbol{u}}+\mathbf{C}\boldsymbol{u}=\mathbf{0}$$

右辺は
ゼロ・ベクトル

有限要素式

$$\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{C}\boldsymbol{u}(t) = \mathbf{0}$$





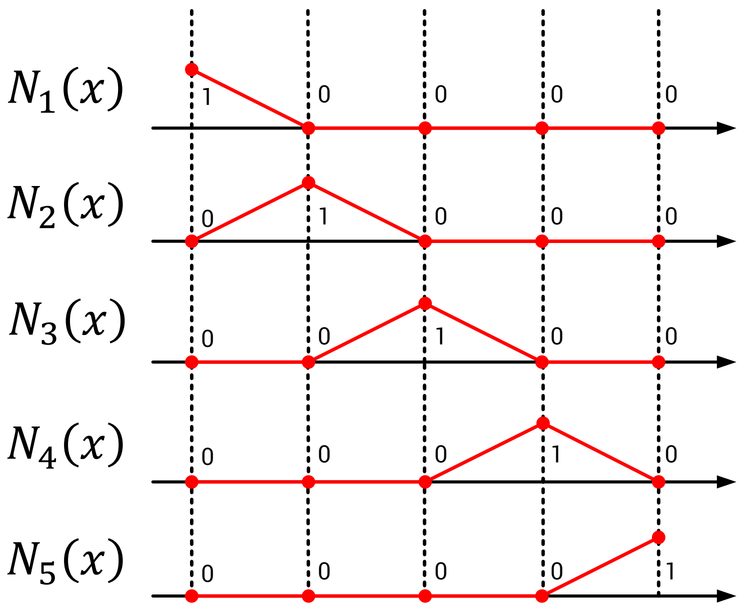
節点(ノード)

$$x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5$$

要素(エレメント)

$$e = 1 \quad 2 \quad 3 \quad 4$$

$$
\begin{aligned}
[\mathbf{M}]_{ij} &= \int_{x_1}^{x_5} N_i(x) N_j(x)\mathrm{d}x \\
&= \int_{x_1}^{x_2} N_i(x) N_j(x)\mathrm{d}x + \int_{x_2}^{x_3} N_i(x) N_j(x)\mathrm{d}x + \int_{x_3}^{x_4} N_i(x) N_j(x)\mathrm{d}x + \int_{x_4}^{x_5} N_i(x) N_j(x)\mathrm{d}x
\end{aligned}
$$

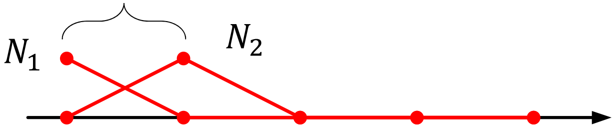


折れ線は要素毎に
直線の式を定義すれば良い

ほとんどの区間でゼロになる

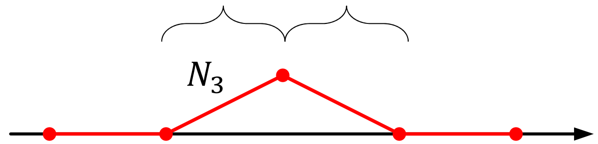
要素iは節点x_iとx_{i+1}の間にあって
$N_i N_i$、$N_i N_{i+1} = N_{i+1} N_i$、$N_{i+1} N_{i+1}$
しか値を持たない！

ここだけ値を持つ

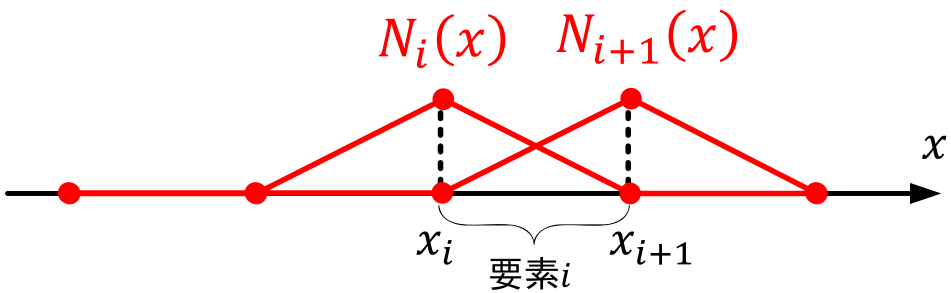


$$\mathbf{M} = \begin{bmatrix} \int_{x_1}^{x_2} N_1 N_1 \mathrm{d}x + 0 + 0 + 0 & \boxed{\int_{x_1}^{x_2} N_1 N_2 \mathrm{d}x + 0 + 0 + 0} & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ \int_{x_1}^{x_2} N_2 N_1 \mathrm{d}x + 0 + 0 + 0 & \int_{x_1}^{x_2} N_2 N_2 \mathrm{d}x + \int_{x_2}^{x_3} N_2 N_2 \mathrm{d}x + 0 + 0 & 0 + \int_{x_2}^{x_3} N_2 N_3 \mathrm{d}x + 0 + 0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0 + \int_{x_2}^{x_3} N_3 N_2 \mathrm{d}x + 0 + 0 & \boxed{0 + \int_{x_2}^{x_3} N_3 N_3 \mathrm{d}x + \int_{x_3}^{x_4} N_3 N_3 \mathrm{d}x + 0} & 0 + 0 + \int_{x_3}^{x_4} N_3 N_4 \mathrm{d}x + 0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0 + 0 + \int_{x_3}^{x_4} N_4 N_3 \mathrm{d}x + 0 & 0 + 0 + \int_{x_3}^{x_4} N_4 N_4 \mathrm{d}x + \int_{x_4}^{x_4} N_4 N_4 \mathrm{d}x & 0 + 0 + 0 + \int_{x_4}^{x_4} N_4 N_5 \mathrm{d}x \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0 + 0 + 0 + \int_{x_4}^{x_4} N_5 N_4 \mathrm{d}x & 0 + 0 + 0 + \int_{x_4}^{x_4} N_5 N_5 \mathrm{d}x \end{bmatrix}$$

ふたつの区間で値を持つ



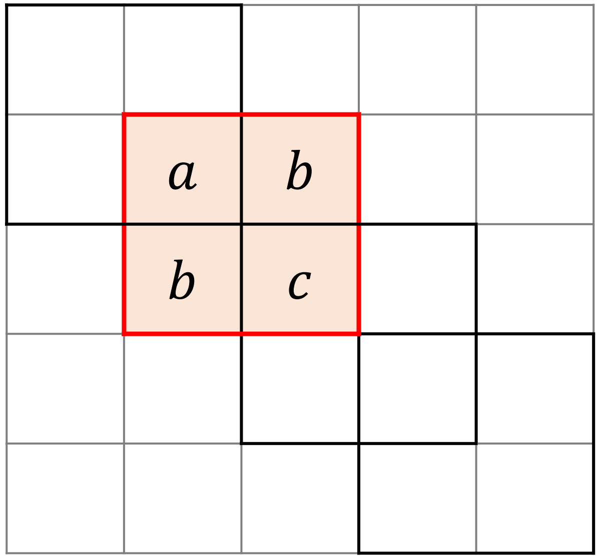
$$\mathbf{M}=\begin{bmatrix}\int_{x_1}^{x_2}N_1N_1\mathrm{d}x & \int_{x_1}^{x_2}N_1N_2\mathrm{d}x & & & \\ \int_{x_1}^{x_2}N_2N_1\mathrm{d}x & \int_{x_1}^{x_2}N_2N_2\mathrm{d}x+\int_{x_2}^{x_3}N_2N_2\mathrm{d}x & \int_{x_2}^{x_3}N_2N_3\mathrm{d}x & & \\ & \int_{x_2}^{x_3}N_3N_2\mathrm{d}x & \int_{x_2}^{x_3}N_3N_3\mathrm{d}x+\int_{x_3}^{x_4}N_3N_3\mathrm{d}x & \int_{x_3}^{x_4}N_3N_4\mathrm{d}x & \\ & & \int_{x_3}^{x_4}N_4N_3\mathrm{d}x & \int_{x_3}^{x_4}N_4N_4\mathrm{d}x+\int_{x_4}^{x_5}N_4N_4\mathrm{d}x & \int_{x_4}^{x_5}N_4N_5\mathrm{d}x \\ & & & \int_{x_4}^{x_5}N_5N_4\mathrm{d}x & \int_{x_4}^{x_5}N_5N_5\mathrm{d}x\end{bmatrix}$$

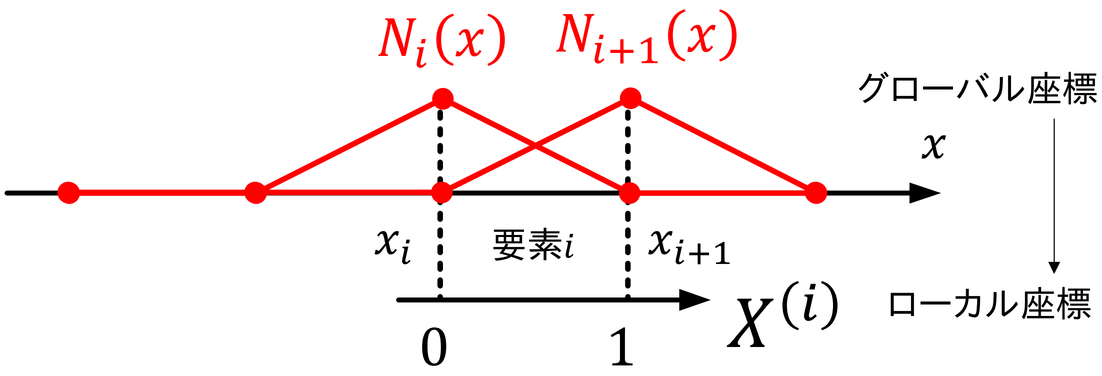


$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$



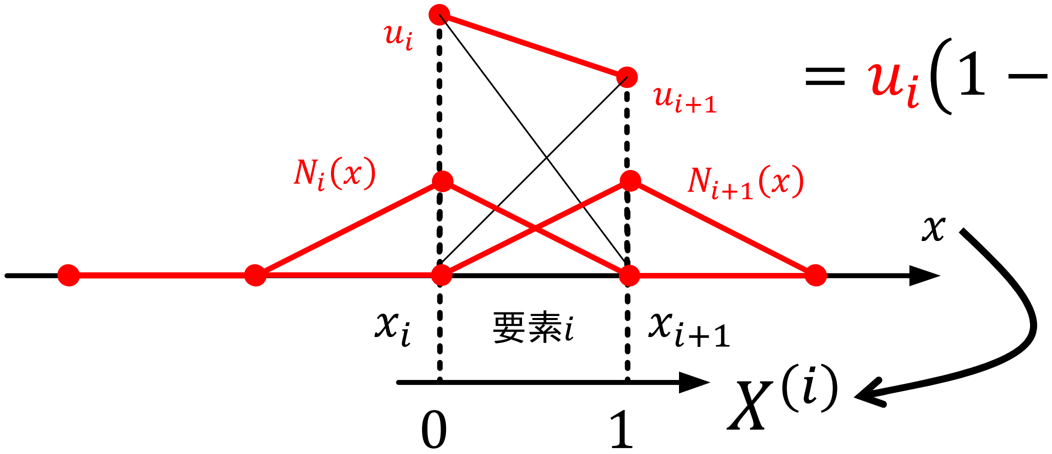


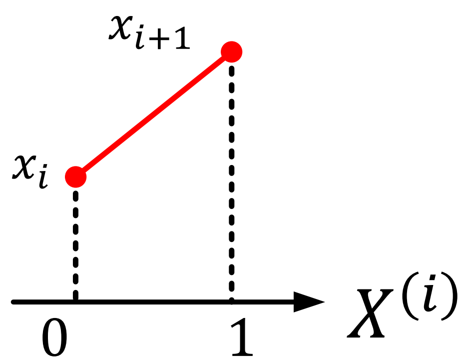
$$N_i(x) \Rightarrow N_1^{(i)}\left(X^{(i)}\right) = 1 - X^{(i)}$$

$$N_{i+1}(x) \Rightarrow N_2^{(i)}\left(X^{(i)}\right) = X^{(i)}$$

$$u_i N_i(x) + u_{i+1} N_{i+1}(x) = u_i N_1^{(i)}\left(X^{(i)}\right) + u_{i+1} N_2^{(i)}\left(X^{(i)}\right)$$

$$= u_i\left(1 - X^{(i)}\right) + u_{i+1} X^{(i)}$$

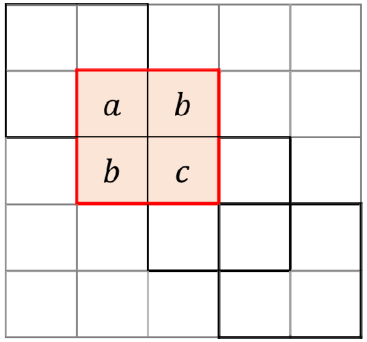




$$N_i(x) \Rightarrow N_1^{(i)}\left(X^{(i)}\right) = 1 - X^{(i)}$$

$$N_{i+1}(x) \Rightarrow N_2^{(i)}\left(X^{(i)}\right) = X^{(i)}$$

$$\boxed{x = x_i\left(1 - X^{(i)}\right) + x_{i+1} X^{(i)}}$$



$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$

$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

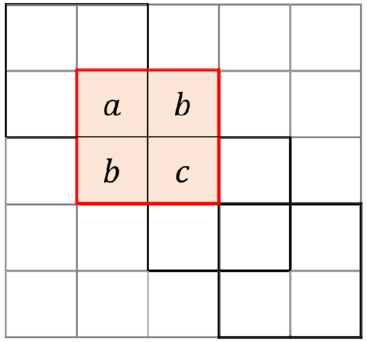
$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}X^{(i)}} \mathrm{d}X^{(i)} = \frac{\mathrm{d}}{\mathrm{d}X^{(i)}} \left(x_i N_1^{(i)}\left(X^{(i)}\right) + x_{i+1} N_2^{(i)}\left(X^{(i)}\right) \right) \mathrm{d}X^{(i)}$$

$$= \frac{\mathrm{d}}{\mathrm{d}X^{(i)}} \left(x_i \left(1 - X^{(i)}\right) + x_{i+1} X^{(i)} \right) \mathrm{d}X^{(i)} = (-x_i + x_{i+1}) \mathrm{d}X^{(i)}$$

$$= (x_{i+1} - x_i) \mathrm{d}X^{(i)} = \Delta x \ \mathrm{d}X^{(i)}$$

$$= \Delta x \int_0^1 N_1^{(i)}\left(X^{(i)}\right) N_1^{(i)}\left(X^{(i)}\right) \mathrm{d}X^{(i)}$$



$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x \; \mathrm{d}X^{(i)}$$

$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

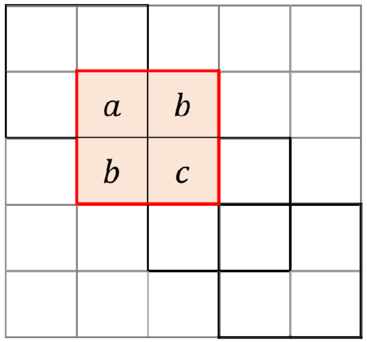
$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$

$$= \Delta x \int_0^1 N_1^{(i)}\left(X^{(i)}\right) N_1^{(i)}\left(X^{(i)}\right) \mathrm{d}X^{(i)}$$

$$= \Delta x \int_0^1 \left(1 - X^{(i)}\right)^2 \mathrm{d}X^{(i)}$$

$$= \Delta x \left[X - X^2 + \frac{1}{3} X^3 \right]_0^1 = \frac{\Delta x}{3}$$



$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

$N_i(x) = N_1^{(i)}\left(X^{(i)}\right) \qquad N_{i+1}(x) = N_2^{(i)}\left(X^{(i)}\right)$

$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)}$

$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

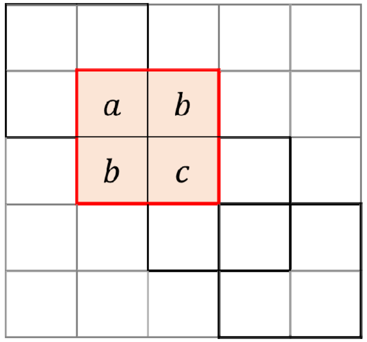
$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$

$$= \Delta x \int_0^1 N_1^{(i)}\left(X^{(i)}\right) N_2^{(i)}\left(X^{(i)}\right) \mathrm{d}X^{(i)}$$

$$= \Delta x \int_0^1 \left(1 - X^{(i)}\right) X^{(i)} \mathrm{d}X^{(i)}$$

$$= \Delta x \left[\frac{1}{2} X^2 - \frac{1}{3} X^3\right]_0^1 = \frac{\Delta x}{6}$$



$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$

$$N_{i+1}(x) = N_2^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)}$$

$$a = \int_{x_i}^{x_{i+1}} N_i(x) N_i(x) \mathrm{d}x$$

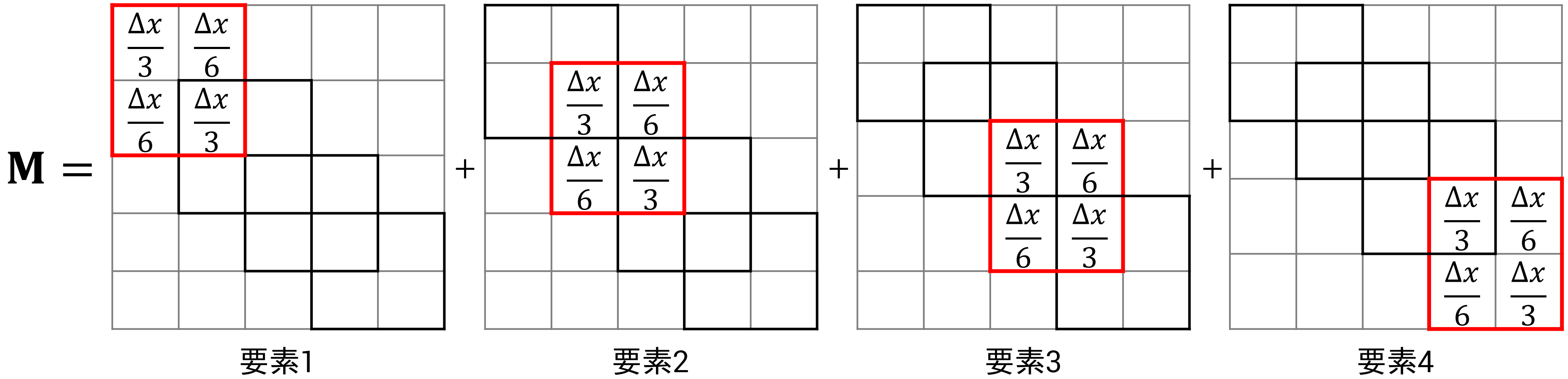
$$b = \int_{x_i}^{x_{i+1}} N_i(x) N_{i+1}(x) \mathrm{d}x$$

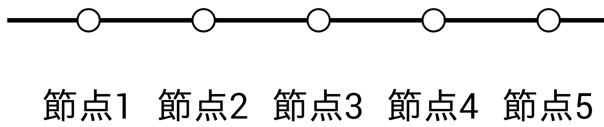
$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) N_{i+1}(x) \mathrm{d}x$$

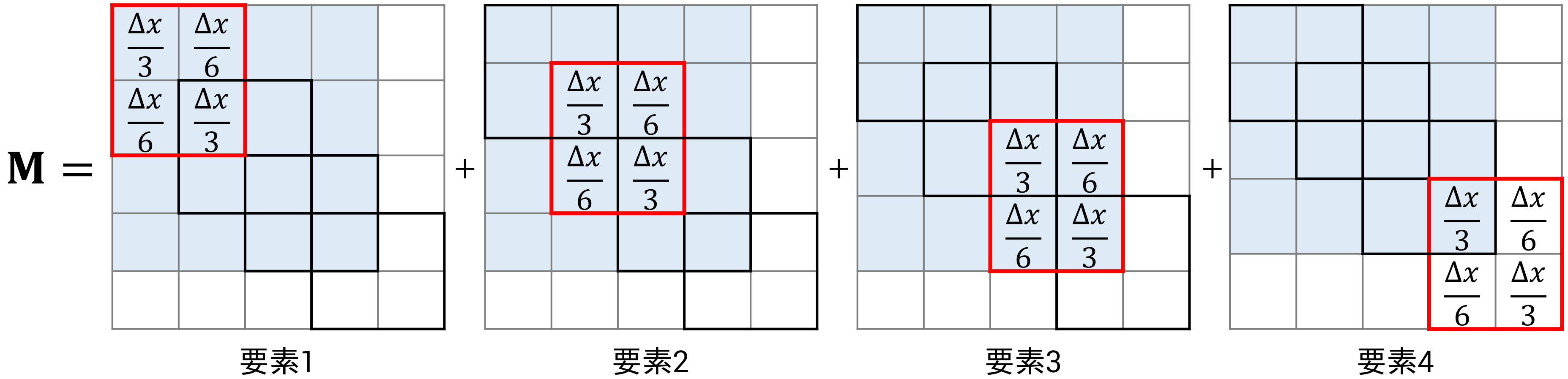
$$= \Delta x \int_0^1 N_2^{(i)}\left(X^{(i)}\right) N_2^{(i)}\left(X^{(i)}\right) \mathrm{d}X^{(i)}$$

$$= \Delta x \int_0^1 \left(X^{(i)}\right)^2 \mathrm{d}X^{(i)}$$

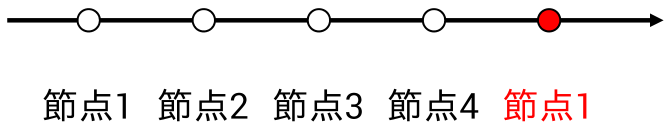
$$= \Delta x \left[\frac{1}{3} X^3\right]_0^1 = \frac{\Delta x}{3}$$


M =
Δx/3 Δx/6
Δx/6 Δx/3
+
+
+
要素1
要素2
要素3
要素4


節点1
節点2
節点3
節点4
節点5



周期境界



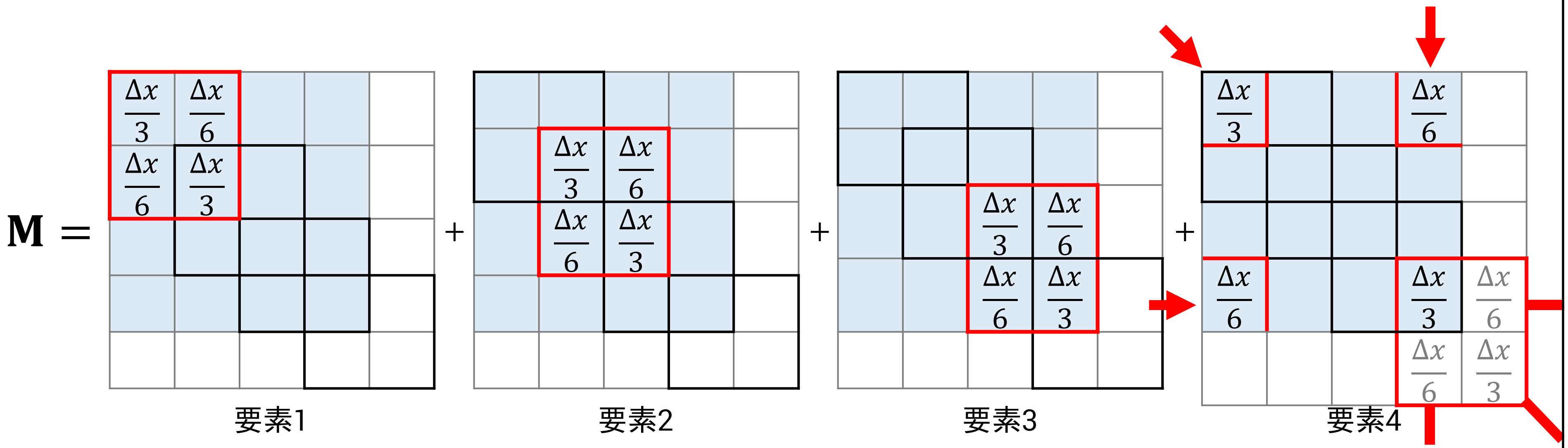
端部の値を共有

$$u_1 = u_5$$

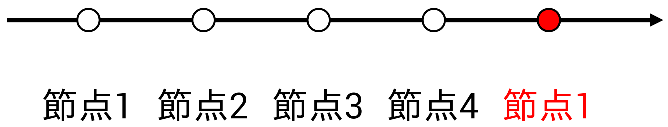
$$N_4 N_5 = N_4 N_1$$

$$N_5 N_4 = N_1 N_4$$

$$N_5 N_5 = N_1 N_1$$



周期境界



$$N_4N_5 = N_4N_1$$

$$N_5N_4 = N_1N_4$$

$$N_5N_5 = N_1N_1$$

端部の値を共有

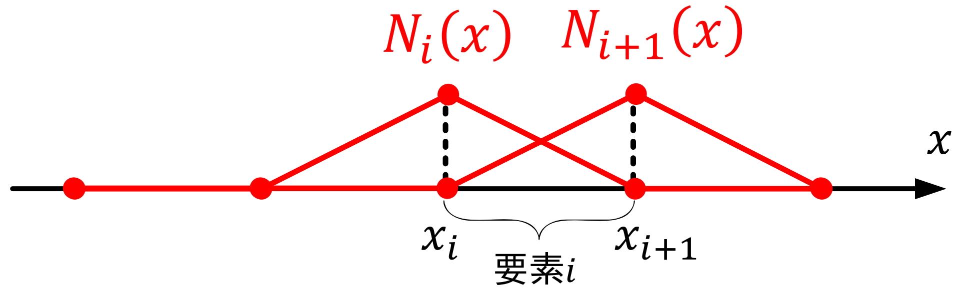
$$u_1 = u_5$$

有限要素式

$$\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{C}\boldsymbol{u}(t) = \mathbf{0}$$

$$\begin{bmatrix} \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & & & \frac{\Delta x}{6} \\ \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & & \\ & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & \\ & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & \\ & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & \\ & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & \\ & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} \\ \frac{\Delta x}{6} & & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} \end{bmatrix} \dot{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}} \quad \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}$$

$$\mathbf{C} = c\begin{bmatrix} \int_{x_1}^{x_2} N_1 \frac{\partial N_1}{\partial x}\mathrm{d}x & \int_{x_1}^{x_2} N_1 \frac{\partial N_2}{\partial x}\mathrm{d}x & & & \\ \int_{x_1}^{x_2} N_2 \frac{\partial N_1}{\partial x}\mathrm{d}x & \int_{x_1}^{x_2} N_2 \frac{\partial N_2}{\partial x}\mathrm{d}x + \int_{x_2}^{x_3} N_2 \frac{\partial N_2}{\partial x}\mathrm{d}x & \int_{x_2}^{x_3} N_2 \frac{\partial N_3}{\partial x}\mathrm{d}x & & \\ & \int_{x_2}^{x_3} N_3 \frac{\partial N_2}{\partial x}\mathrm{d}x & \int_{x_2}^{x_3} N_3 \frac{\partial N_3}{\partial x}\mathrm{d}x + \int_{x_3}^{x_4} N_3 \frac{\partial N_3}{\partial x}\mathrm{d}x & \int_{x_3}^{x_4} N_3 \frac{\partial N_4}{\partial x}\mathrm{d}x & \\ & & \int_{x_3}^{x_4} N_4 \frac{\partial N_3}{\partial x}\mathrm{d}x & \int_{x_3}^{x_4} N_4 \frac{\partial N_4}{\partial x}\mathrm{d}x + \int_{x_4}^{x_5} N_4 \frac{\partial N_4}{\partial x}\mathrm{d}x & \int_{x_4}^{x_5} N_4 \frac{\partial N_5}{\partial x}\mathrm{d}x \\ & & & \int_{x_4}^{x_5} N_5 \frac{\partial N_4}{\partial x}\mathrm{d}x & \int_{x_4}^{x_5} N_5 \frac{\partial N_5}{\partial x}\mathrm{d}x \end{bmatrix}$$

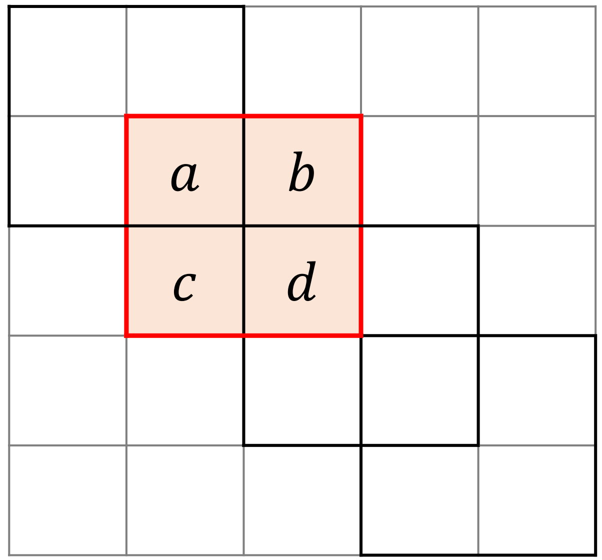


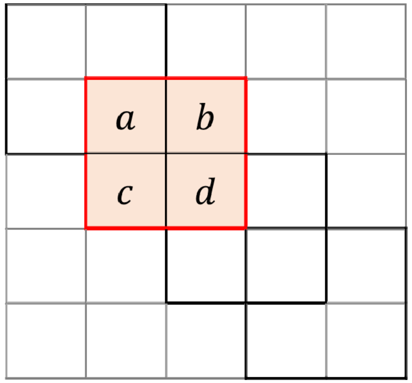
$$a = \int_{x_i}^{x_{i+1}} N_i(x) \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} N_i(x) \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$d = \int_{x_i}^{x_{i+1}} N_{i+1}(x) \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$





$$a = \int_{x_i}^{x_{i+1}} N_i(x) \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} N_i(x) \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$d = \int_{x_i}^{x_{i+1}} N_{i+1}(x) \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

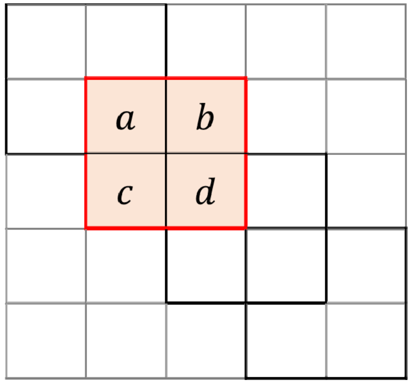
$$a = \int_{x_i}^{x_{i+1}} N_i(x) \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \boxed{\Delta x}\, \mathrm{d}X^{(i)}$$

$$\frac{\partial N_i(x)}{\partial x} = \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \frac{\partial X^{(i)}}{\partial x} = \frac{1}{\left(\frac{\partial x}{\partial X^{(i)}}\right)} \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} = \boxed{\frac{1}{\Delta x}} \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}$$

$$= \int_0^1 N_1^{(i)}\left(X^{(i)}\right) \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \mathrm{d}X^{(i)}$$



$$a = \int_{x_i}^{x_{i+1}} N_i(x)\frac{\partial N_i(x)}{\partial x}\mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} N_i(x)\frac{\partial N_{i+1}(x)}{\partial x}\mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x)\frac{\partial N_i(x)}{\partial x}\mathrm{d}x$$

$$d = \int_{x_i}^{x_{i+1}} N_{i+1}(x)\frac{\partial N_{i+1}(x)}{\partial x}\mathrm{d}x$$

$$a = \int_{x_i}^{x_{i+1}} N_i(x)\frac{\partial N_i(x)}{\partial x}\mathrm{d}x$$

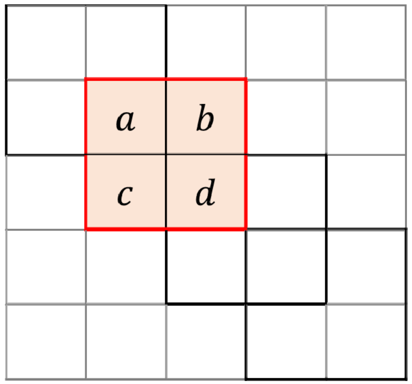
$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x\,\mathrm{d}X^{(i)} \qquad \frac{\partial N_i(x)}{\partial x} = \frac{1}{\Delta x}\frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}$$

$$= \int_0^1 N_1^{(i)}\left(X^{(i)}\right)\frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}\mathrm{d}X^{(i)}$$

$$= \int_0^1 \left(1 - X^{(i)}\right)(-1)\mathrm{d}X^{(i)}$$

$$= \left[-X + \frac{1}{2}X^2\right]_0^1 = -\frac{1}{2}$$



$$a = \int_{x_i}^{x_{i+1}} N_i(x) \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} N_i(x) \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$d = \int_{x_i}^{x_{i+1}} N_{i+1}(x) \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} N_i(x) \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

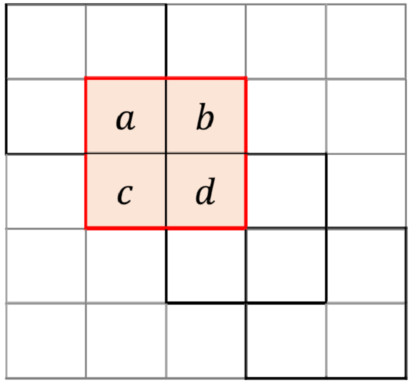
$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right) \qquad N_{i+1}(x) = N_2^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)} \qquad \frac{\partial N_i(x)}{\partial x} = \frac{1}{\Delta x} \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}$$

$$= \int_0^1 N_1^{(i)}\left(X^{(i)}\right) \frac{\partial N_2^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \mathrm{d}X^{(i)}$$

$$= \int_0^1 \left(1 - X^{(i)}\right)(1)\mathrm{d}X^{(i)}$$

$$= \left[X - \frac{1}{2}X^2\right]_0^1 = \frac{1}{2}$$



$$a = \int_{x_i}^{x_{i+1}} N_i(x) \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$b = \int_{x_i}^{x_{i+1}} N_i(x) \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

$$d = \int_{x_i}^{x_{i+1}} N_{i+1}(x) \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) \frac{\partial N_i(x)}{\partial x} \mathrm{d}x$$

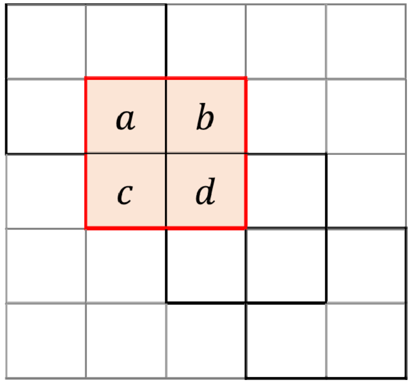
$$N_i(x) = N_1^{(i)}\left(X^{(i)}\right) \qquad N_{i+1}(x) = N_2^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)} \qquad \frac{\partial N_i(x)}{\partial x} = \frac{1}{\Delta x} \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}$$

$$= \int_0^1 N_2^{(i)}\left(X^{(i)}\right) \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \mathrm{d}X^{(i)}$$

$$= \int_0^1 \left(X^{(i)}\right)(-1)\mathrm{d}X^{(i)}$$

$$= \left[-\frac{1}{2}X^2\right]_0^1 = -\frac{1}{2}$$



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$$c = \int_{x_i}^{x_{i+1}} N_{i+1}(x) \frac{\partial N_{i+1}(x)}{\partial x} \mathrm{d}x$$

$$N_{i+1}(x) = N_2^{(i)}\left(X^{(i)}\right)$$

$$\mathrm{d}x = \Delta x \, \mathrm{d}X^{(i)} \qquad \frac{\partial N_i(x)}{\partial x} = \frac{1}{\Delta x} \frac{\partial N_1^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}}$$

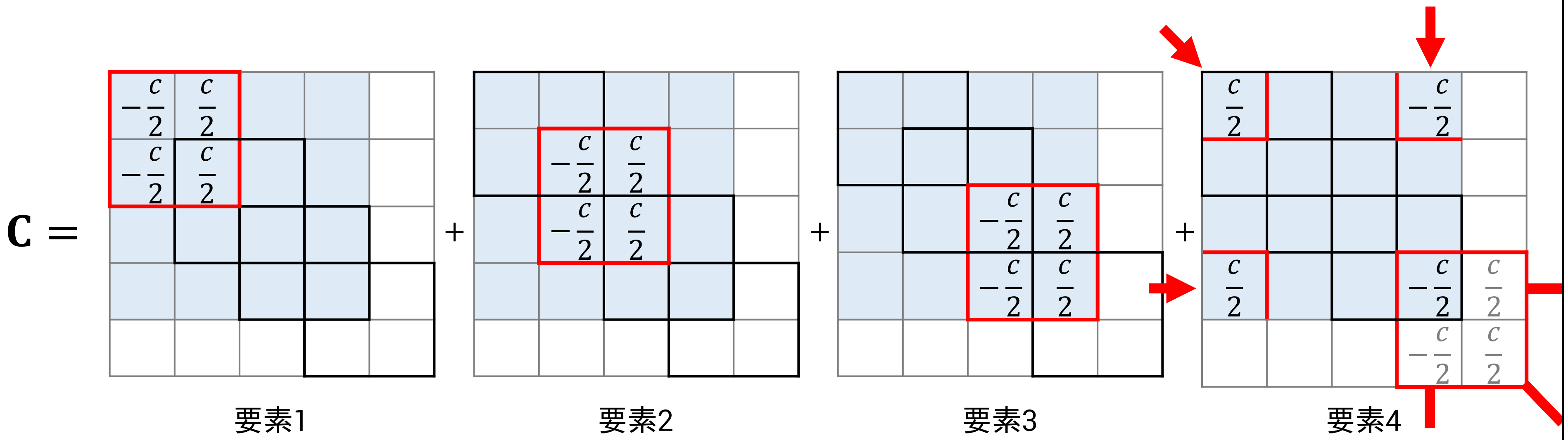
$$= \int_0^1 N_2^{(i)}\left(X^{(i)}\right) \frac{\partial N_2^{(i)}\left(X^{(i)}\right)}{\partial X^{(i)}} \mathrm{d}X^{(i)}$$

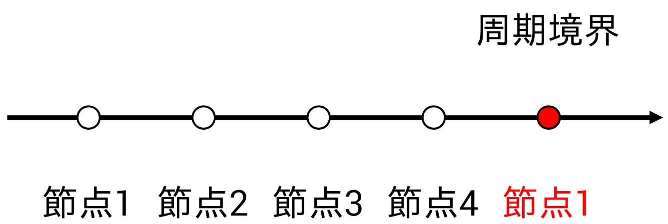
$$= \int_0^1 \left(X^{(i)}\right)(1)\mathrm{d}X^{(i)}$$

$$= \left[\frac{1}{2} X^2\right]_0^1 = \frac{1}{2}$$

$$
\mathbf{C}=\underbrace{\begin{bmatrix} -\frac{c}{2} & \frac{c}{2} & & & \\ -\frac{c}{2} & \frac{c}{2} & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}}_{\text{要素1}}+\underbrace{\begin{bmatrix} & & & & \\ & -\frac{c}{2} & \frac{c}{2} & & \\ & -\frac{c}{2} & \frac{c}{2} & & \\ & & & & \\ & & & & \end{bmatrix}}_{\text{要素2}}+\underbrace{\begin{bmatrix} & & & & \\ & & & & \\ & & -\frac{c}{2} & \frac{c}{2} & \\ & & -\frac{c}{2} & \frac{c}{2} & \\ & & & & \end{bmatrix}}_{\text{要素3}}+\underbrace{\begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & -\frac{c}{2} & \frac{c}{2} \\ & & & -\frac{c}{2} & \frac{c}{2} \end{bmatrix}}_{\text{要素4}}
$$

節点1　節点2　節点3　節点4　節点5





$$N_4 N_5 = N_4 N_1$$

$$N_5 N_4 = N_1 N_4$$

$$N_5 N_5 = N_1 N_1$$

端部の値を共有

$$u_1 = u_5$$

有限要素式

$$\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{C}\boldsymbol{u}(t) = \mathbf{0}$$

$$\begin{bmatrix}
\frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & & & \frac{\Delta x}{6} \\
\frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & & \\
 & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & \\
 & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & \\
 & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & \\
 & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & \\
 & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} \\
\frac{\Delta x}{6} & & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3}
\end{bmatrix}
\begin{bmatrix} \dot{u}_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}
\qquad
\begin{bmatrix}
0 & \frac{c}{2} & & & & & & -\frac{c}{2} \\
-\frac{c}{2} & 0 & \frac{c}{2} & & & & & \\
 & -\frac{c}{2} & 0 & \frac{c}{2} & & & & \\
 & & -\frac{c}{2} & 0 & \frac{c}{2} & & & \\
 & & & -\frac{c}{2} & 0 & \frac{c}{2} & & \\
 & & & & -\frac{c}{2} & 0 & \frac{c}{2} & \\
 & & & & & -\frac{c}{2} & 0 & \frac{c}{2} \\
\frac{c}{2} & & & & & & -\frac{c}{2} & 0
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}$$

前進差分法

$$\dot{\boldsymbol{u}}(t) = \frac{\boldsymbol{u}(t+\Delta t) - \boldsymbol{u}(t)}{\Delta t}$$

計算不可は小さいが
不安定な場合が多い

後退差分法

$$\dot{\boldsymbol{u}}(t) = \frac{\boldsymbol{u}(t) - \boldsymbol{u}(t-\Delta t)}{\Delta t}$$

計算不可は大きいが
安定な場合が多い

$$\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{C}\boldsymbol{u}(t) = \mathbf{0}$$

後退差分法

$$\boxed{\dot{\boldsymbol{u}}(t) = \frac{\boldsymbol{u}(t) - \boldsymbol{u}(t - \Delta t)}{\Delta t}}$$

$$\mathbf{M}\frac{\boldsymbol{u}(t) - \boldsymbol{u}(t - \Delta t)}{\Delta t} + \mathbf{C}\boldsymbol{u}(t) = \mathbf{0}$$

有限要素式

$$\boxed{\mathbf{M}\dot{\boldsymbol{u}}(t) + \mathbf{C}\boldsymbol{u}(t) = \mathbf{0}}$$

$$\mathbf{M}\frac{\boldsymbol{u}(t) - \boldsymbol{u}(t - \Delta t)}{\Delta t} + \mathbf{C}\boldsymbol{u}(t) = \mathbf{0}$$

$$\mathbf{M}\big(\boldsymbol{u}(t) - \boldsymbol{u}(t - \Delta t)\big) + \Delta t\mathbf{C}\boldsymbol{u}(t) = \mathbf{0}$$

$$(\mathbf{M} + \Delta t\mathbf{C})\boldsymbol{u}(t) = \mathbf{M}\boldsymbol{u}(t - \Delta t)$$

グローバルマトリクス　解ベクトル　右辺ベクトル

連立方程式
$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$
のイメージ

$$\boxed{\boldsymbol{u}(t) = [(\mathbf{M} + \Delta t\mathbf{C})^{-1}\mathbf{M}]\boldsymbol{u}(t - \Delta t)}$$

シミュレーション・コードの計算式

シミュレーション・コードの計算式

$$\boldsymbol{u}(t) = [(\mathbf{M} + \Delta t\mathbf{C})^{-1}\mathbf{M}]\,\boldsymbol{u}(t - \Delta t)$$

① $\mathbf{M}$マトリクスを作成する

② $\mathbf{C}$マトリクスを作成する

③ グローバルマトリクス：$\mathbf{A}$を作成する
$\mathbf{A} = (\mathbf{M} + \Delta t\mathbf{C})$

④ 初期値$\boldsymbol{u}(0)$を設定する

⑤ $n = 1$

⑥ 時刻$t = n\Delta t$の数値解$\boldsymbol{u}(n\Delta t)$を求める

⑦ $\mathrm{n} = n + 1$

⑧ 繰り返す

MATLABコーディング

$\boldsymbol{u}$の初期値設定

```
u = [zeros(100,1); ones(300,1); zeros(600,1)];
```

$\boldsymbol{u}$の長さは要素数(=節点数)　※周期境界だから

```
n = length(u);
```

その他、パラメータを設定する

```
c = 1.0;
dt = 0.0001;
t = 0:dt:(2*1/c);
T = length(t);
dx= 1/n;
x = 0:dx:(n-1)*dx;
```

```
t = 0:dt:(2*1/c)
```

0 から (2*1/c)
まで刻みをdtで
数列を生成する

```
x = 0:dx:(n-1)*dx
```

0 から (n-1)*dx
まで刻みをdxで
数列を生成する

MATLABコーディング

$$
\begin{array}{c|cccccccc|}
 & 1 & & & ii & & & & n \\
\hline
1 & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & & & \frac{\Delta x}{6} \\
 & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & & \\
 & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & & \\
ii & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & & \\
 & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & & \\
 & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} & \\
 & & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} & \frac{\Delta x}{6} \\
n & \frac{\Delta x}{6} & & & & & & \frac{\Delta x}{6} & \frac{2\Delta x}{3} \\
\hline
\end{array}
$$

全ての成分がゼロのn×n行列を作成

```
M = zeros(n,n);
```

（初期化）

```
M = zeros(n,n);
for ii=1:n
    if ii==1
        M(ii,n)    = dx*1/6;
        M(ii,ii)   = dx*2/3;
        M(ii,ii+1)= dx*1/6;
    elseif ii==n
        M(ii,ii-1)= dx*1/6;
        M(ii,ii)   = dx*2/3;
        M(ii,1)    = dx*1/6;
    else
        M(ii,ii-1)= dx*1/6;
        M(ii,ii)   = dx*2/3;
        M(ii,ii+1)= dx*1/6;
    end
end
```

MATLABコーディング

0	$\frac{c}{2}$						$-\frac{c}{2}$
$-\frac{c}{2}$	0	$\frac{c}{2}$					
	$-\frac{c}{2}$	0	$\frac{c}{2}$				
		$-\frac{c}{2}$	0	$\frac{c}{2}$			
			$-\frac{c}{2}$	0	$\frac{c}{2}$		
				$-\frac{c}{2}$	0	$\frac{c}{2}$	
					$-\frac{c}{2}$	0	$\frac{c}{2}$
$\frac{c}{2}$						$-\frac{c}{2}$	0

```
C = zeros(n,n);
for ii=1:n
    if ii==1
        C(ii,n)    = -c/2;
        C(ii,ii+1)= c/2;
    elseif ii==n
        C(ii,ii-1)= -c/2;
        C(ii,1)    = c/2;
    else
        C(ii,ii-1)= -c/2;
        C(ii,ii+1)= c/2;
    end
end
```

シミュレーション・コードの計算式

$$\boldsymbol{u}(t) = \underbrace{[(\mathbf{M} + \Delta t\mathbf{C})^{-1}\mathbf{M}]}\,\boldsymbol{u}(t - \Delta t)$$

```
A = (M+dt*C)¥M;
```

バックスラッシュ

¥は＼

左を逆行列にして
右に掛ける、の意

初期化

```
u = [u zeros(n,T-1)];
```

```
for tt=1:T-1
    u(:,tt+1)=A*u(:,tt);
end
```

時刻ttの時の$\boldsymbol{u}$を用いて
時刻tt+1の時の$\boldsymbol{u}$を求める

MATLABコーディング

結果のVisualization

Figure領域に
時刻ttのグラフを描画

動画ファイルの
フレームに保存

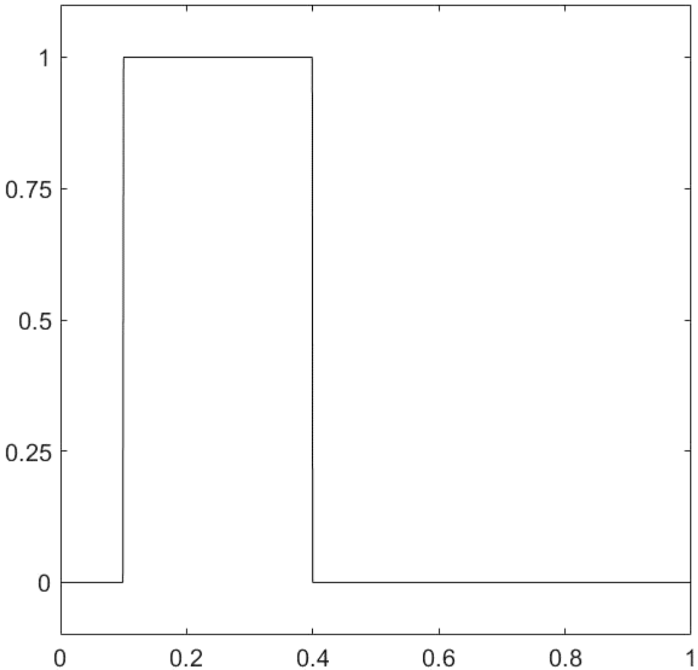
時刻ttを更新

```
% open file
[y,m,d]    = ymd(datetime);
ymdstr     = [num2str(y) num2str(m,'%0.2d') num2str(d,'%0.2d')];
filename   = ['ex_01_' ymdstr '.avi'];
writerObj = VideoWriter(filename,'Motion JPEG AVI');
writerObj.FrameRate = 20; %-- flame rate (slide/sec)
open(writerObj); %-- initialization of movie object

% avi visualization
figure(1)
Fnum=100;
for tt=1:round(T/Fnum):T
    set(gcf, 'position', [100   100   480   480 ])
    set(gcf, 'color',[1 1 1])
    plot((0:dx:1)',[u2(:,tt);u2(1,tt)],'k-')
    set(gca, 'position', [0.10  0.08  0.87  0.87])
    set(gca, 'fontname', 'Arial', 'fontsize', 12)
    xlim([0 1])
    ylim([-0.1 1.1])
    set(gca, 'YTick', -1:0.25:1)
    frame = getframe(figure(1)); %-- get frame
    writeVideo(writerObj,frame); %-- add frame to video obj
end

close(writerObj);
```

解析解



数値解