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FINITE ELEMENT ANALYSIS OF QUASI-STATIC AND DYNAMIC COLLAPSE BEHAVIORS OF FRAMED STRUCTURES BY THE ADAPTIVELY SHIFTED INTEGRATION TECHNIQUE

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Abstract The adaptively shifted integration technique (abbreviated to the ASI technique) is applied to the dynamic collapse problems of framed structures including the quasi-static collapse under repeated loading. The unloading of materials plays an important role in these behaviors. In the present analysis using cubic beam elements, the reshifting of the integration points to the Gaussian points in the unloaded elements is conducted in order to attain higher accuracy for the material-unloading behavior. The solutions given by conventional method, the ASI technique with and without reshifting are compared to show the validity of the newly proposed computational algorithm.

1. INTRODUCTION

The new computational algorithm identified as the adaptively shifted integration technique has been developed by the present authors as a smart algorithm for the nonlinear finite element analysis of framed structures and has been applied to the static collapse problems such as plastic collapse and elasto-plastic buckling collapse problems [1–3]. In the nonlinear finite element analysis by the ASI technique, the highest computational efficiency for large-scale frame analysis has been achieved by shifting the numerical integration points for the evaluation of stiffness matrices (i.e. strain energy) of the linear Timoshenko beam element [4] or the cubic beam element based on Bernoulli–Euler hypothesis [5], depending upon the elastic-plastic property of the element. It is noted that the ASI technique can be easily implemented in the existing codes.

The ASI technique is applied to the dynamic collapse problems in the present study. The numerical integration points shifted with the formation of a plastic hinge [6] were not shifted, even if the element returned to the elastic state with the occurrence of unloading, since the unloading behavior was not predominant in the plastic collapse and the elasto-plastic buckling problems under static loading treated in the previous reports [2, 3]. However, an accurate simulation for the loading-unloading behavior accompanied by the elasto-plastic wave propagation is important in the analysis of dynamic collapse problems. The unloading behavior plays an important role also in the static collapse problem under repeated loading. The present paper describes a smart algorithm by the ASI technique for the dynamic collapse problems, including the static collapse under repeated loading considering the reshifting in an unloaded element and verifies its validity by the numerical studies for the shake-down, the incremental collapse and the dynamic collapse problems of simple framed structures.

2. ASI TECHNIQUE FOR LOADING-UNLOADING BEHAVIORS OF FRAMED STRUCTURES

The ASI technique is applicable to two types of finite elements which are the linear Timoshenko beam element and the cubic beam element based on Bernoulli–Euler hypothesis. The present paper deals with the cubic element. However, it should be noted that a similar procedure is also applicable to the linear element.

In the cubic beam element based on the Bernoulli–Euler hypothesis, the relation between the locations of the two numerical integration points s_i $(-1 \leqslant s_i \leqslant 1)$ and those of the stress evaluation points or of the occurrence of plastic hinges r_i $(-1 \leqslant r_i \leqslant 1)$ is expressed by the following eqns [1]:

$$r_i = \mp \frac{1}{3s_2} \quad (i = 1, 2;\ s_1 = -s_2) \tag{1a}$$

or

$$s_i = \mp \frac{1}{3r_2} \quad (i = 1, 2;\ r_1 = -r_2), \tag{1b}$$

where $s_i = \mp 1/3r_2$ means $s_1 = -1/3r_2$ and $s_2 = 1/3r_2$ and the rest is similarly defined. The above relation was obtained by considering the equivalent condition for the strain energy approximation between the cubic beam element and the rigid bodies-spring model in computational discontinuum mechanics [1]. When the Gaussian integration is used (i.e. $s_i = \mp 1/\sqrt{3}$), plastic hinges are formed at the same points. When $s_i = \mp 1/3$, plastic hinges can exactly occur at the edge points of the element.

When the entire region in an element behaves elastically, the Gaussian integration points are located at the optimal positions which give the most accurate linear solutions. From eqn (1), $r_i = \mp 1/\sqrt{3}$ in this case. The incremental stiffness equation for the element is then expressed as

$$[k]\{\Delta u\} = \{\Delta f\}, \tag{2a}$$

where

$$[k] = \frac{L}{2}\left\{\left[B\left(-\frac{1}{\sqrt{3}}\right)\right]^t\left[D_e\left(-\frac{1}{\sqrt{3}}\right)\right]\left[B\left(-\frac{1}{\sqrt{3}}\right)\right] + \left[B\left(\frac{1}{\sqrt{3}}\right)\right]^t\left[D_e\left(\frac{1}{\sqrt{3}}\right)\right]\left[B\left(\frac{1}{\sqrt{3}}\right)\right]\right\}. \tag{2b}$$

In the equations, the following notations are used: $[k]$ is the tangential stiffness matrix, $\{\Delta u\}$ is the nodal displacement increment vector, $\{\Delta f\}$ is the nodal external force increment vector, $[B(s)]$ is the generalized strain-nodal displacement matrix, $[D(r)]$ is the elastic resultant force-generalized strain matrix, L is the element length. The generalized strain increment vector is calculated as

$$\left\{\Delta\epsilon\left(\pm\frac{1}{\sqrt{3}}\right)\right\} = \left[B\left(\pm\frac{1}{\sqrt{3}}\right)\right]\{\Delta u\}. \tag{3}$$

The resultant force increment vector is then evaluated as

$$\left\{\Delta\sigma\left(\pm\frac{1}{\sqrt{3}}\right)\right\} = \left[D_e\left(\pm\frac{1}{\sqrt{3}}\right)\right]\left\{\Delta\epsilon\left(\pm\frac{1}{\sqrt{3}}\right)\right\}. \tag{4}$$

The distributions of the resultant forces are given by the following form of equation:

$$\{\Delta\sigma(s)\} = [T_1(s)]\left\{\Delta\sigma\left(-\frac{1}{\sqrt{3}}\right)\right\} + [T_2(s)]\left\{\Delta\sigma\left(\frac{1}{\sqrt{3}}\right)\right\}. \tag{5}$$

The position of the cross-section in the element which will first reach a fully-plastic state can be determined by the following equations:

$$f(\sigma(-r_2)) = 0 \tag{6a}$$

or

$$f(\sigma(r_2)) = 0, \tag{6b}$$

where $f(\sigma(r))$ is a yield function expressed in terms of the resultant forces.

Immediately after the occurrence of the fully-plastic section, the numerical integration points are shifted to the new positions

$$s_i = \mp\frac{1}{3r_2}$$

according to eqn (1) so as to form a plastic hinge exactly at the position of the fully-plastic section. If a fully-plastic section has occurred at either of the end points of the element (i.e. $r_i = \mp 1$), the numerical integration points are moved to $s_i = \mp 1/3$. Assuming that the fully-plastic section has first occurred at $s = r_2(>0)$, the incremental stiffness equation at the following incremental step is then expressed as

$$[k]\{\Delta u\} = \{\Delta f\}, \tag{7a}$$

where

$$[k] = \frac{L}{2}\left\{\left[B\left(-\frac{1}{3r_2}\right)\right]^t[D_e(-r_2)]\left[B\left(-\frac{1}{3r_2}\right)\right] + \left[B\left(\frac{1}{3r_2}\right)\right]^t[D_p(r_2)]\left[B\left(\frac{1}{3r_2}\right)\right]\right\}. \tag{7b}$$

The generalized strain increment vector is calculated as

$$\{\Delta\epsilon(\pm r_2)\} = \left[B\left(\pm\frac{1}{3r_2}\right)\right]\{\Delta u\}. \tag{8}$$

The resultant force increment vector is evaluated as

$$\{\Delta\sigma(-r_2)\} = [D_e(-r_2)]\{\Delta\epsilon(-r_2)\} \tag{9a}$$

$$\{\Delta\sigma(r_2)\} = [D_p(r_2)]\{\Delta\epsilon(r_2)\}. \tag{9b}$$

The resultant force increments at the new integration points ($\{\Delta\sigma(\pm r_2)\}$ calculated by eqns (8) and (9)) are physically those at $s = \pm r_2$, which, therefore, must be added to the resultant forces calculated at $s = \pm r_2$ before yielding ($\{\sigma(\pm r_2)\}$ determined by eqns (5) and (6)). After the other integration point $(-s_1)$ has yielded, the stress matrix $[D_e(-r_2)]$ in eqns (7b) and (9a) is replaced with $[D_p(-r_2)]$.

The above mentioned procedure is the computational algorithm of the ASI technique which has been used for the behavior without unloading or in the case when the effect of the unloading is secondary. The following algorithm must be added in the analysis of the quasi-static collapse behavior under repeated loading and the dynamic collapse behavior.

When all of the plastic hinges formed are judged to have been unloaded by the following conditions, the calculation continues, reshifting the numerical integration points to the Gaussian points at the following step:

$$\lfloor \frac{\partial f}{\partial \sigma}(-r_2) \rfloor [D_e(-r_2)] \{\Delta\epsilon(-r_2)\} < 0 \quad (10a)$$

$$\lfloor \frac{\partial f}{\partial \sigma}(r_2) \rfloor [D_e(r_2)] \{\Delta\epsilon(r_2)\} < 0. \quad (10b)$$

That is, eqns (2a) and (2b) are used as an element stiffness equation. The generalized strain increments and the resultant force increments are calculated by eqns (3) and (4). The resultant force increments at an arbitary position, including the point of the plastic hinge formation, are evaluated by eqn (5), and the reyielding is judged by eqn (6). The computational procedure after reyielding follows again the equations starting from eqn (7).

The dynamic collapse analysis is made possible by adding inertial force terms to the quasi-static computational procedure, including reshifting of the numerical integration points as mentioned above. That is, the following equation of motion for the complete system is integrated by an appropriate time integration scheme:

$$[M]\{\ddot{u}\}_{t+\Delta t} + [K]\{\Delta u\} = \{F\}_{t+\Delta t} - \{R\}_t. \quad (11)$$

where the following notations are used; $[M]$ is the mass matrix for the complete system, $[K]$ is the stiffness matrix for the complete system, $\{F\}_{t+\Delta t}$ is the nodal external force vector at $t+\Delta t$, $\{R\}_t$ is the nodal internal force vector at t, $\{\ddot{u}\}_{t+\Delta t}$ is the nodal acceleration vector at $t+\Delta t$, $\{\Delta u\}$ is the nodal displacement increment vector between t and $t+\Delta t$. The present analysis employs consistent mass as an element mass matrix which is assumed to not be influenced by the shifting of the numerical integration points. Newmark's β method ($\beta = 1/4, \gamma = 1/2$) [7] is used as a numerical integration scheme for eqn (11).

The incremental analysis proceeds with an optimum modeling for the whole process of the elastic behavior before initial yielding, the loading behavior after yielding and the elastic unloading behavior in the present computational procedure, which is expected to be the most effective computational algorithm for the quasi-static analysis under repeated loading and the dynamic collapse analysis. The concrete forms of the equations in the above-described formulation is given in Refs [1, 2].

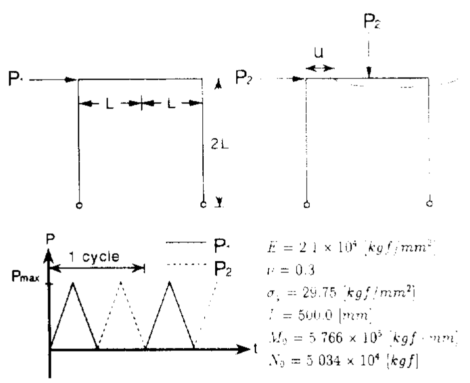

Fig. 1. Plane frame under repeated loading.

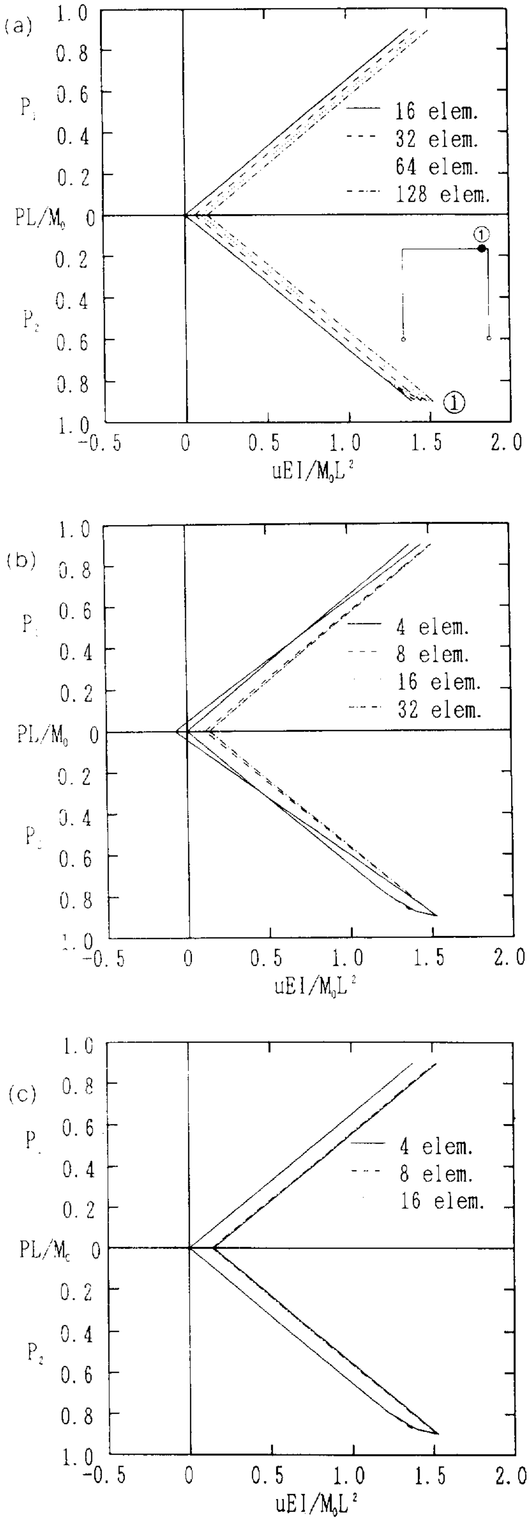

Fig. 2. Load-displacement relationships in shake-down analysis: (a) conventional method; (b) ASI technique without reshifting; (c) ASI technique with reshifting.

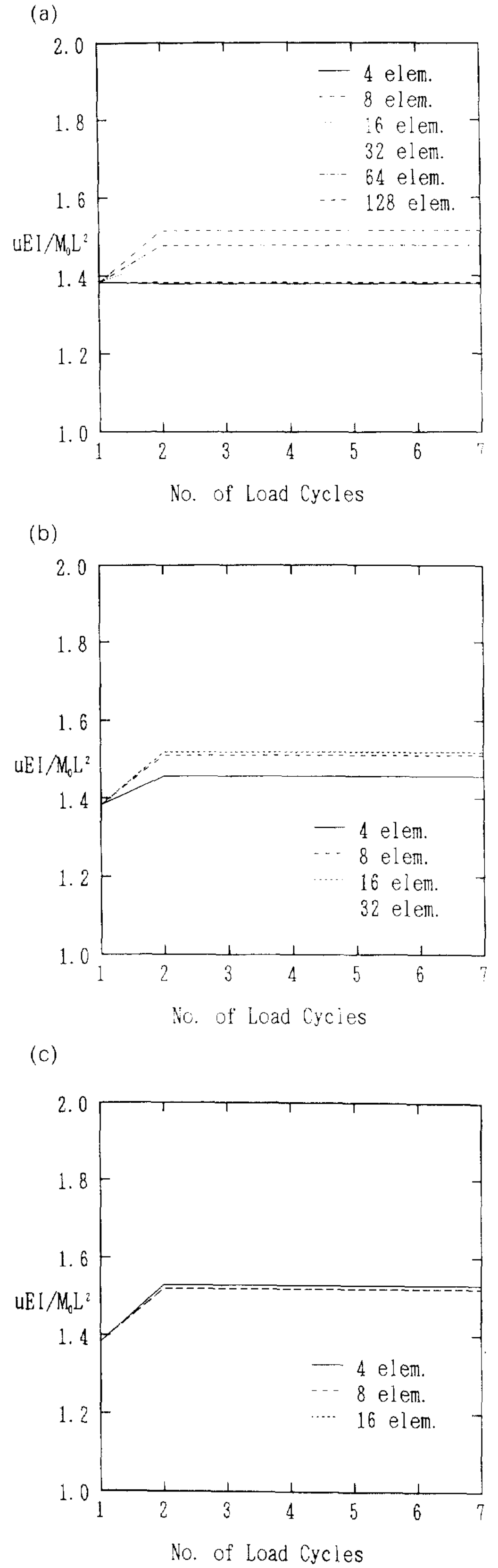

Fig. 3. Displacement vs loading cycles in shake-down analysis: (a) conventional method; (b) ASI technique without reshifting; (c) ASI technique with reshifting.

3. SHAKE-DOWN AND INCREMENTAL COLLAPSE ANALYSIS OF FRAMED STRUCTURES

Figure 1 shows the analyzed problems [6]. The portal frame with pin-jointed lower ends is subjected to the horizontal loading P_1 at the left, upper edge and the vertical loading P_2 at the midspan of the beam alternately. There occur two cases of the shake-down ($P_{max} = 0.90M_0/L$) and the incremental collapse ($P_{max} = 0.96M_0/L$), depending on the common maximum value to both loads P_{max}. Each case has been analyzed by the following three methods in order to show the validity of the ASI technique in comparison with the conventional method: (a) the conventional finite element method in which the numerical integration points are fixed to the Gaussian integration points; (b) the ASI technique without reshifting of the numerical integration points at unloading; and (c) the ASI technique with reshifting of the numerical integration points (to the Gaussian integration points) at unloading.

3.1. *Shake-down analysis*

Figures 2 and 3 show the computed results in which there has occurred the shake-down. The load–displacement curves and the displacement-loading cycles relations are shown in Figs 2 and 3, respectively. Note that P_1 and P_2 are plotted in the upper and the lower directions, respectively, on the vertical axis of Fig. 2. The displacement in Figs 2 and 3 is the horizontal displacement at the left, upper edge (see Fig. 1). The numbers of elements shown in the figures are those for the whole structure. The beam has two elements and each column has one element in the four-element subdivision, which is the minimum possible number of elements. The bisection of each element gives successively eight-element, 16-element and 32-element subdivisions.

As shown in Fig. 3a, the conventional method has not given the shake-down behavior but the elastic deformation in the cases of four-, eight- and 16-element subdivisions. The convergence of solutions is extremely slow. These results are due to the use of the computational algorithm in which stresses are evaluated and plastic hinges are formed only at the Gaussian integration points. On the other hand, the ASI technique without reshifting has successfully simulated the shake-down phenomenon, even by the four-element modeling and the convergence of solutions is fast, as shown in Fig. 3b. However, it is seen that there is room for improvement in the elastic rigidity at unloading in the case of fewer elements. This is because the numerical integration points have been shifted to $s_i = \mp 1/3$ and the calculation has continued without reshifting at unloading. The improved ASI technique with reshifting of the numerical integration points to the Gaussian points has given fairly accurate solutions even by the coarse mesh.

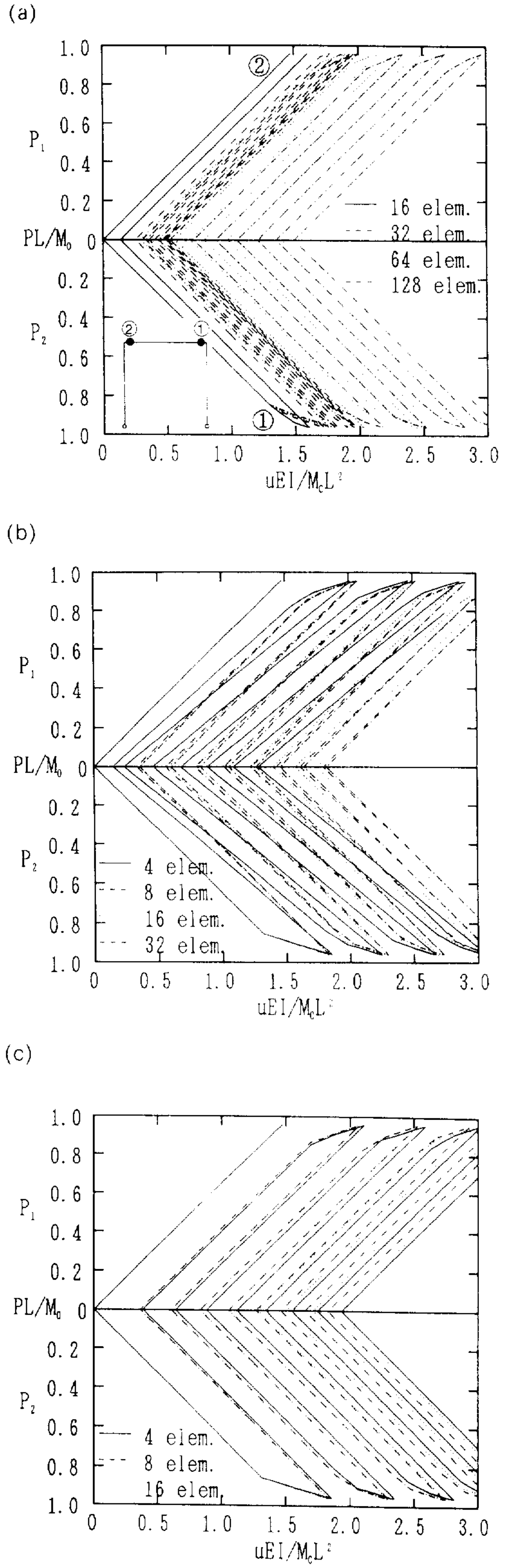

Fig. 4. Load–displacement relationships in incremental collapse analysis: (a) conventional method; (b) ASI technique without reshifting; (c) ASI technique with reshifting.

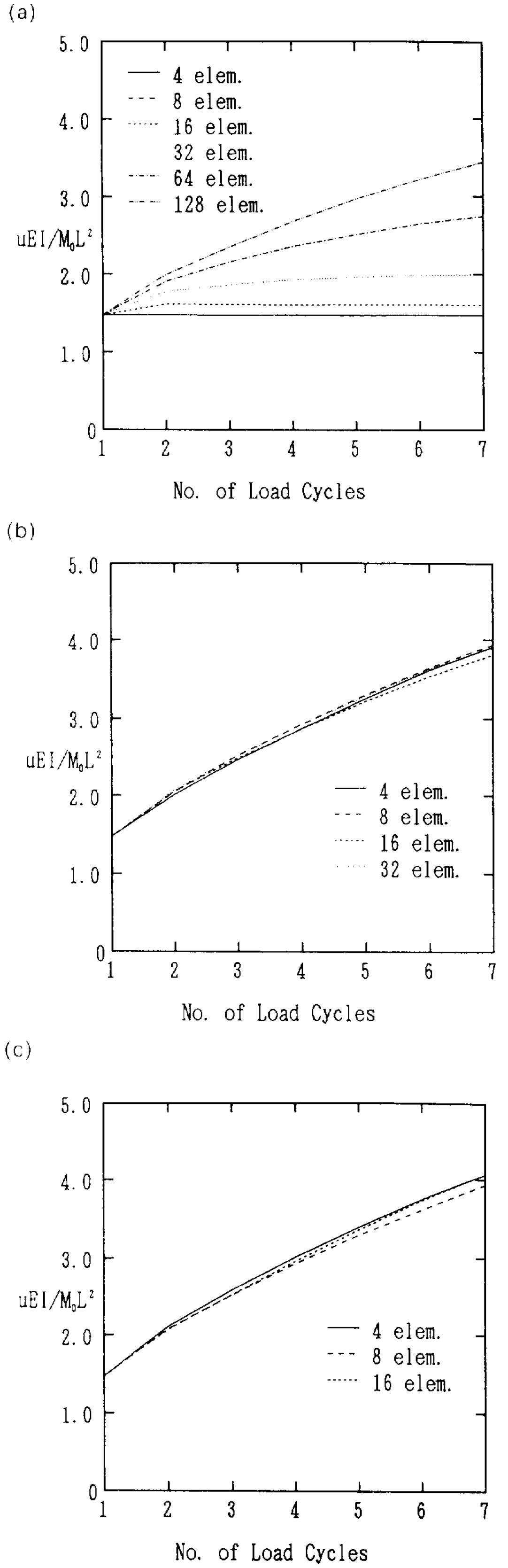

Fig. 5. Displacement vs loading cycles in incremental collapse analysis: (a) conventional method; (b) ASI technique without reshifting; (c) ASI technique with reshifting.

3.2. *Incremental collapse analysis*

Figures 4 and 5 are the computed results in which there has occurred the incremental collapse. The loading–displacement curves and the relationship between displacement and loading cycles are plotted in Figs 4 and 5, respectively. The numbers of elements shown in the figures have the same meaning as in the shake-down analysis.

As shown in Fig. 5a, the conventional method has given the shake-down like behaviors in the case of coarse meshes such as the four- and eight-element subdivisions. The convergence of solutions is extremely slow. These results are due to the fact that stresses are evaluated and plastic hinges are formed at the Gaussian integration points. On the other hand, the ASI technique has considerably improved the accuracy and the convergency of solutions. The four-element modeling has given a practically sufficient solution. The displacements at the end of each loading cycle as shown in Figs 5b and c are almost permanent deformations due to plastic deformations, therefore they are not much influenced by the reshifting at unloading. However, comparing Figs 4b and c, it is clear that the elastic rigidity at unloading in the load–displacement curves has been considerably improved by the reshifting at unloading.

Figure 6 shows the results by the non-uniform meshes as is often used. The 10-element approximations for the whole structure have been used, in which smaller elements are placed in the neighborhood of member joints and a concentratedly loaded point. The solutions depend a lot on the length of the smaller elements (they converge to the exact solution as a limit of zero length!) as seen from the three results of cases 1–3. This method is not easy to use.

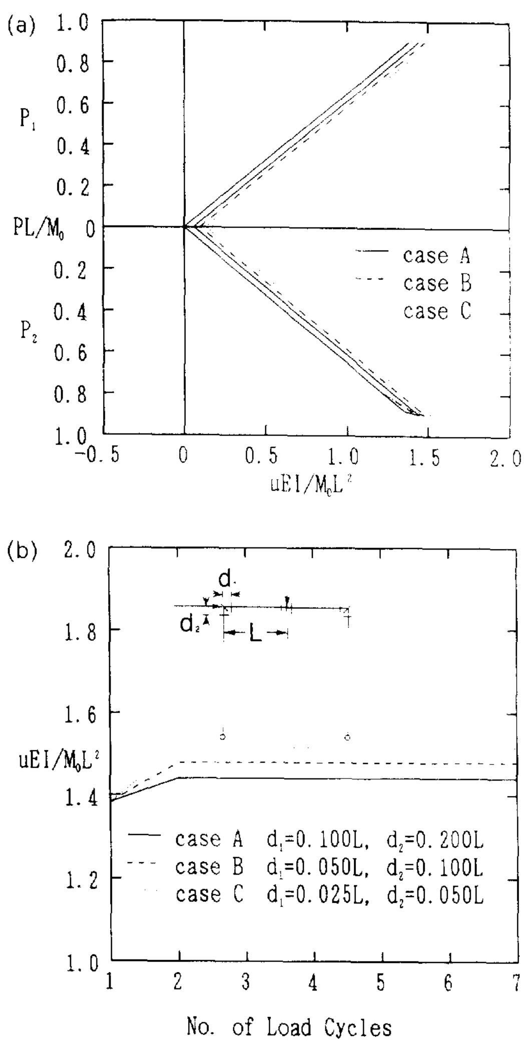

Fig. 6. Shake-down analysis by the conventional method with non-uniform mesh, (a) load–displacement relationships and (b) displacement vs loading cycles.

4. DYNAMIC COLLAPSE ANALYSIS OF FRAMED STRUCTURES

The dynamic collapse behavior of a simple plane and a space frame subjected to step loading is analyzed in the present section. Each frame is analyzed by the following three methods in order to show the validity of the ASI technique in comparison with the conventional method: (a) the conventional finite element

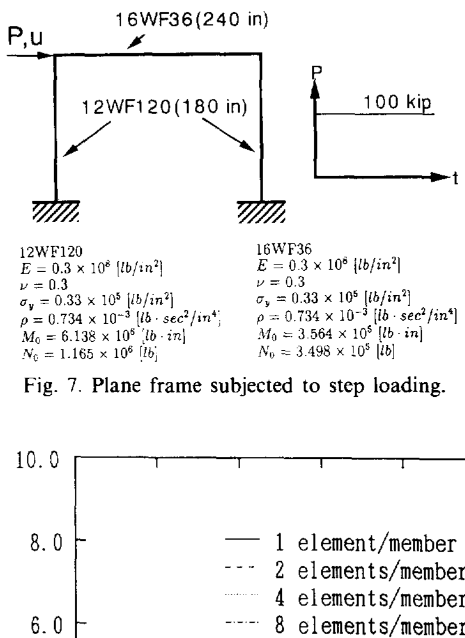

Fig. 7. Plane frame subjected to step loading.

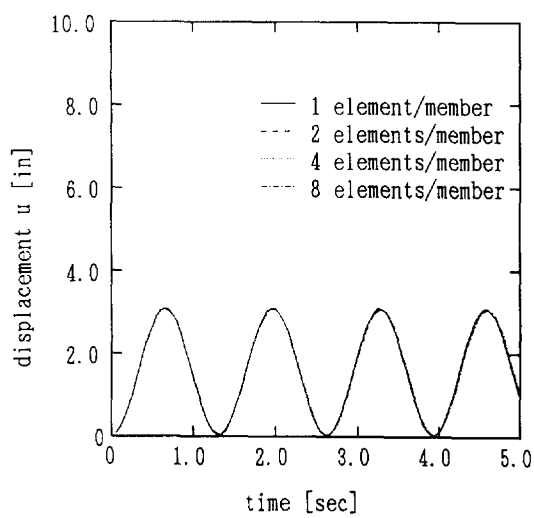

Fig. 8. Elastic response of a plane frame.

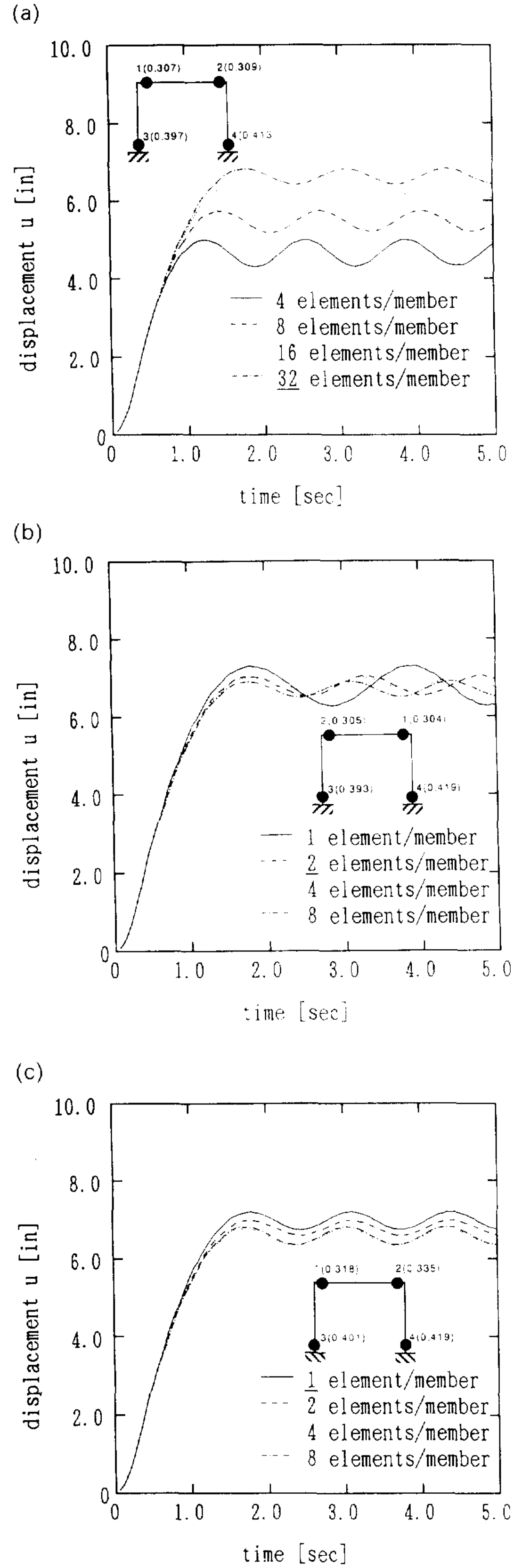

Fig. 9. Elasto-plastic response of a plane frame: (a) conventional method; (b) ASI technique without reshifting; (c) ASI technique with reshifting.

method in which the numerical integration points are fixed to the Gaussian integration points; (b) the ASI technique without reshifting of the numerical integration points at unloading; and (c) the ASI technique with reshifting of the numerical integration points (to the Gaussian integration points) at unloading.

4.1. *Dynamic collapse analysis of a plane frame*

Figure 7 shows the analyzed problem [8]. The portal frame with fixed lower ends is subjected to a horizontal step loading at the left, upper edge. The mass of steel members shown in the figure was multiplied by 625 in the analysis.

Figure 8 is the result of the elastic response analysis (there is no distinction between the ASI technique and the conventional method in the elastic analysis). It is seen that the one-element modeling per member has given a converged solution in the present problem, not including the vibration modes of higher order.

Figure 9 shows the time histories of the horizontal displacement at the loaded point given by the three methods. In the result of Fig. 9a, given by the conventional method, the convergence is extremely slow as in the quasi-static analysis under repeated loading. The result of Fig. 9b, given by the ASI technique without reshifting of the numerical integration points is considerably improved in comparison with the solution by the conventional method. However, there is a lot of error in the stationary vibration period in the case of fewer elements. This is due to the fact that the stationary elastic vibration accompanied by the unloading behavior cannot be analyzed accurately by the algorithm, which does not reshift the numerical integration points from $s_i = \mp 1/3$. The convergency is good, both with respect to the absolute value of displacement and the vibration period in the results of Fig. 9c given by the ASI technique, with reshifting the numerical integration points at unloading. The one-element modeling per

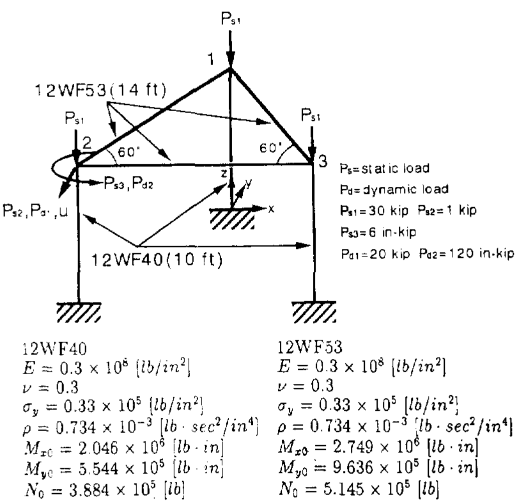

Fig. 10. Space frame subjected to step loading.

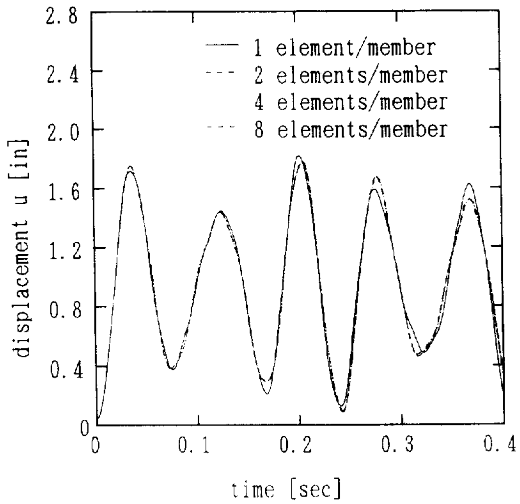

Fig. 11. Elastic response of a space frame.

member can practically give sufficient solutions in the present problem, not including the vibration modes of higher order.

The computed results for the time sequence of occurrence of plastic hinges by the underlined number of elements are also shown in Fig. 9. It is also seen from these results that the one-element per member solution by the ASI technique with reshifting of integration points is comparable to the result with the 32-element approximation by the conventional method.

4.2. *Dynamic collapse analysis of a space frame*

Figure 10 shows the analyzed example [9]. The triangular-shaped rigid frame with fixed lower ends is subjected to a dead load (self-weight) and a step loading at the member joint 2. Figure 11 is the result of the elastic response analysis. It is seen that the one-element subdivision has given the solution which almost agrees with the converged solution.

Figure 12 shows the time histories of the displacement in the x direction at the loaded point (the point 2) given by the three methods. The results are qualitatively similar to those for the plane frame in the preceding subsection. That is, the convergence is so slow that even the 16-element per member solution includes several percentage of error in the results of Fig. 12a given by the conventional method. The results of Fig. 12b, given by the ASI technique without reshifting of the numerical integration points, are more accurate than those by the conventional method. However, there is much error with respect to the vibration period in the fewer-element solutions. Both the absolute value of the displacement and the vibration period are quite accurate in the results given by the ASI technique with reshifting, which can improve the accuracy for the elastic vibration accompanied by the unloading, as shown in Fig. 12c.

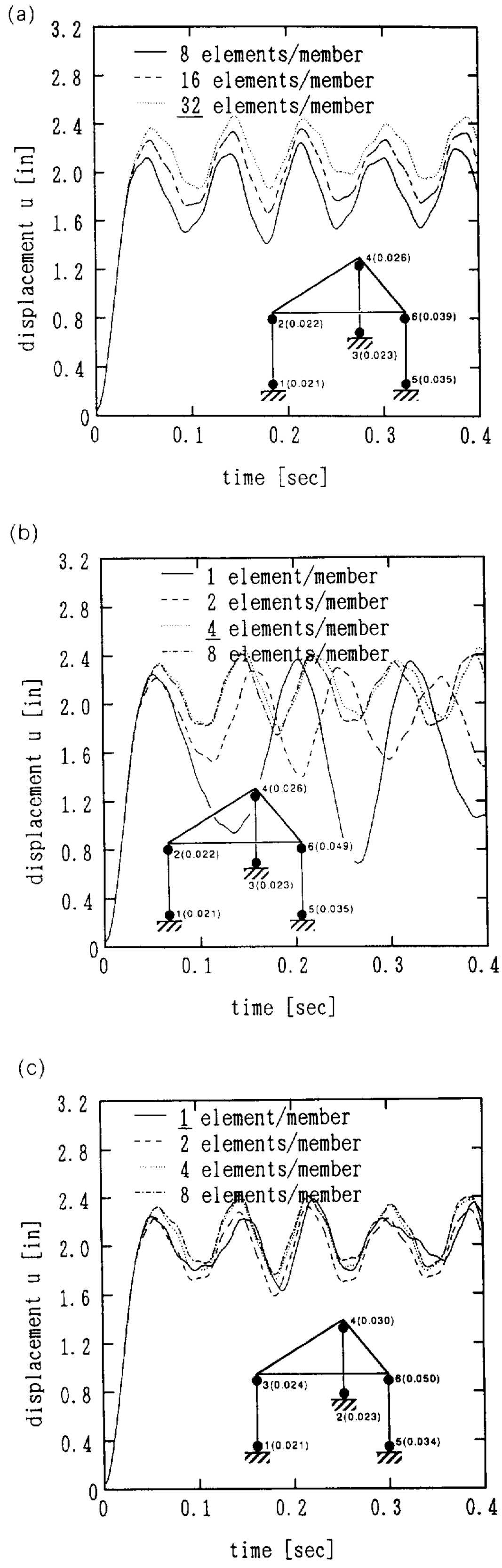

Fig. 12. Elasto-plastic response of a space frame: (a) conventional method; (b) ASI technique without reshifting; (c) ASI technique with reshifting.

It is seen that one-element per member modeling can give practically sufficient solutions in the present problem.

The computed results for the time sequence of occurrence of plastic hinges by the underlined number of elements are also shown in Fig. 12. It is seen that the two-element per member solution by the ASI technique with reshifting of the integration points well corresponds to the result with the 32-element approximation by the conventional method.

5. CONCLUDING REMARKS

The ASI technique, which has been proposed as a smart algorithm for the nonlinear finite element analysis of large-scale framed structures, has been applied to the quasi-static collapse problem under repeated loading and the dynamic collapse problem. The reshifting algorithm of the numerical integration points at unloading has been devised, considering the importance of the unloading behavior. As a result, a considerable improvement of computation accuracy has been achieved, not only in comparison with the conventional method but also with the ASI technique without reshifting. The implementation of the ASI technique with reshifting in the existing codes is quite easy.

REFERENCES

1. Y. Toi, Shifted integration technique in one-dimensional plastic collapse analysis using linear and cubic finite elements. *Int. J. numer. Meth. Engng* **31**, 1537–1552 (1991).
2. Y. Toi and D. Isohe, Adaptively shifted integration technique for finite element collapse analysis of framed structures. *Int. J. numer. Meth. Engng* **36**, 2323–2339 (1993).
3. Y. Toi and D. Isobe, Adaptively shifted integration technique for nonlinear finite element analysis of framed structures. In: *Proc. 3rd World Congress on Computational Mechanics*, Makuhari, 1–5, August (1994).
4. T. J. R. Hughes, R. L. Taylor and W. Kanoknukulchai, A simple and efficient finite element for plate bending. *Int. J. numer. Meth. Engng* **11**, 1529–1543 (1977).
5. G. H. Gallagher, *Finite Element Analysis: Fundamentals.* Prentice-Hall, Englewood Cliffs, NJ (1975).
6. P. G. Hodge, Jr, *Plastic Analysis of Structures.* McGraw-Hill, New York (1959).
7. N. M. Newmark, A method of computation for structural dynamics. *Proc. ASCE 85* **EM3**, 67–94 (1959).
8. T. G. Toridis and K. Kozeimeh, Inelastic response of frames to dynamic loads, *Proc. ASCE 97* **EM3**, 847–863 (1971).
9. R. K. Wen and F. Farhoomand, Dynamic analysis of inelastic space frames. *Proc. ASCE 96* **EM5**, 667–686 (1970).